

# Local hypothesis testing between a pure bipartite state and the white noise state

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**Abstract**—In this paper, we treat a local discrimination problem in the framework of asymmetric hypothesis testing. We choose a known bipartite pure state  $|\Psi\rangle$  as an alternative hypothesis, and the completely mixed state as a null hypothesis. As a result, we analytically derive an optimal type 2 error and an optimal POVM for one-way LOCC POVM and Separable POVM. For two-way LOCC POVM, we study a family of simple three-step LOCC protocols, and show that the best protocol in this family has strictly better performance than any one-way LOCC protocol in all the cases where there may exist difference between two-way LOCC POVM and one-way LOCC POVM.

**Index Terms**—Local discrimination, Hypothesis testing, LOCC, Separable Operations.

## I. INTRODUCTION

In all quantum information processings, we always need to measure quantum states in order to derive classical information encoded there. Because of this, since an early stage of the field of quantum information, people have made effort to understand how well a given unknown quantum state can be identified when a set of candidates is given [1], [2]. People deal this problem with different theoretical frameworks in the sub-fields of quantum information named, Quantum State discrimination [3], [5], Quantum hypothesis testing [6], [7], [8], Quantum State Estimation [9], [10], [11], and Classical Capacity of Quantum Channel [12], [13], [14] <sup>1</sup>.

Because of decoherence, we generally need to pay a lot of cost to reliably send a quantum state to a spatially separated place. Thus, it is important to study quantum information processing in a situation where reliable quantum communication is not available across spatially separated places; this restriction for available quantum operations leads a class of quantum operations called LOCC (Local Operations and Classical Communication), and also other slightly different classes of quantum operations like Separable Operations, PPT (Positive Partial Transpose) operations, etc [17], [18], [19], [20]. Thus, many researches have been done to study how well a given partially unknown state can be identified under these restricted quantum operations [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], [52], [53], [54], [55], [56], [57]. These researches are often called researches of “*Local discrimination*”.

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<sup>1</sup>Other references about these topic can be found in the reference lists of [15], [16].

In this paper, we treat local discrimination in the framework of an asymmetric hypothesis testing where we do not use any prior probability on a set of candidates.

In a hypothesis testing, we aim to certify a given hypothesis  $H_1$  (called “*alternative hypothesis*”), and in order to do it, we try to reject a hypothesis  $H_0$  (called “*null hypothesis*”) which is true when  $H_1$  is false. Hence, we try to minimize the error probability judging  $H_0$  to be true when  $H_1$  is true (the type 2 error) under the condition that a fixed value  $\alpha$  upper-bounds the error probability judging  $H_1$  to be true when  $H_0$  is true (the type 1 error). When both  $H_0$  and  $H_1$  consist of a single state, a hypothesis testing looks very similar to a normal state discrimination. However, they are different in the way to treat errors: two kind of errors are treated in a completely asymmetric way in a hypothesis testing, and in a symmetric way in a state discrimination (although their prior may not be symmetric).

The number of researches of an asymmetric quantum hypothesis testing is rather small with respect to that of quantum state discrimination; a partial list of researches of asymmetric quantum hypothesis testing may include [6], [7], [58], [59], [60], [61], [62], [63], [64], [65], [66]. In particular, concerning hypothesis testing with local restrictions (we will call “*local hypothesis testing*” in this paper), only very restricted number of papers treated it [40], [47], [55].

In this paper, we consider the situation where two spatially separated parties detect a signal in a known bipartite pure state  $|\Psi\rangle$ . They try to certify that what they detected is not a noise, but a state  $|\Psi\rangle$ . On this purpose, we choose  $|\Psi\rangle$  as an alternative hypothesis and the completely mixed state, which represents a white noise, as a null hypothesis. As a class of local measurements, we treat one-way LOCC POVM (Positive Operator Valued Measure), two-way LOCC POVM, and Separable POVM [16], [19], [20], [68], [69]. This study can be considered as a generalization of our previous paper [47]; see Section II for detail discussion about their relation.

As a result, we analytically derive an optimal type 2 error and an optimal POVM for one-way LOCC POVM and Separable POVM. In particular, in order to derive an analytical solution for Separable POVM, we prove the equivalence of the local hypothesis testing under Separable POVM and a global hypothesis testing with a composite null hypothesis, and analytically solve this global hypothesis testing. Furthermore, for two-way LOCC POVM, we study a family of simple three-step LOCC protocols, and show that the best protocol in this family has strictly better performance than any one-way LOCC protocol in all the cases where there may exist difference between two-way LOCC POVM and one-way LOCC POVM.

In quantum information, so far, just a very limited number of works treat a hypothesis testing with a composite hypothesis [55], [57], [67]. In this paper, on the way to derive analytical solutions to the local hypothesis testing under separable POVMs, we add one example into this category. Our example consists of a composite null hypothesis and a simple alternative hypothesis on a single partite Hilbert-space. A set of the null hypothesis is generated from a single pure state by phase flipping operations. We give an analytical solution for this global hypothesis testing with a composite null hypothesis.

This paper is organized as follows: We explain notations and problem settings in Section II, and, then, present main results of the paper in Section III. One-way and two-way LOCC are treated in Section IV. We give an analytical solution for a global hypothesis testing with a composite hypothesis in Section V, and then, prove the equivalence between this hypothesis testing and the local hypothesis testing under Separable POVM in Section VI. Finally, we give a summary in Section VII. We also add appendix to present a proof for a corollary.

## II. PRELIMINARY

### A. Notations

First, we introduce our notations. A finite bipartite Hilbert space is called as  $\mathcal{H}_{AB} \stackrel{\text{def}}{=} \mathcal{H}_A \otimes \mathcal{H}_B$ . We define  $d_A, d_B$  and  $d$  as  $d_A \stackrel{\text{def}}{=} \dim \mathcal{H}_A, d_B \stackrel{\text{def}}{=} \dim \mathcal{H}_B$  and  $d \stackrel{\text{def}}{=} \min\{d_A, d_B\}$ , respectively. Normally, we assume that two spatially separated parties, say Alice and Bob, possess these two local Hilbert spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively. The space of all operators on  $\mathcal{H}$  is called  $\mathfrak{B}(\mathcal{H})$ . The space of all Hermitian operators on  $\mathcal{H}$  is called  $\mathcal{P}(\mathcal{H})$ . The cone of all positive operators on  $\mathcal{H}$  is called  $\mathcal{P}_+(\mathcal{H})$ .  $\{a < \rho \leq b\}$  denotes a projection onto a direct sum of eigenspaces whose eigenvalues  $\lambda$  satisfy  $a < \lambda < b$ .

In this paper, we only consider a two-valued POVM  $\{T, I_{AB} - T\}$ ;  $T \in \mathfrak{B}(\mathcal{H})$  satisfies  $0 \leq T \leq I_{AB}$ . Since a two-valued POVM is completely determined by fixing an element  $T$ , we often say ‘‘POVM  $T$ ’’ as an abbreviation of ‘‘POVM  $\{T, I - T\}$ ’’<sup>2</sup> in this paper. A word ‘‘global POVM’’ just means a POVM with no additional restriction, and we denote a set of all two-valued POVMs on  $\mathcal{H}_{AB}$  as  $g$ . A POVM is called a two-way LOCC POVM, if it can be implemented by two-way LOCC (local operations with two-way classical communication) [16], [19], [20], [69].  $\leftrightarrow$  denotes a set of all two-values two-way LOCC. Similarly, a POVM is called a one-way LOCC POVM, if it can be implemented by one-way LOCC (local operations with one-way classical communication) [16], [19], [20], [69]. There are two-different sets of one-way LOCC corresponding to two-different directions of one-way classical communication (C.C.); that is, one-way LOCC with C.C. from Alice to Bob and it with C.C. from Bob to Alice. These two types of one-way LOCC should be treated distinctly. However, in our case, the final results (an optimal error or success probability) corresponding to one set can easily be derived from another

by just swapping the dimension of Alice and Bob. We just treat a set of one-way LOCC POVMs from Alice to Bob, and we write this set as  $\rightarrow$ . A POVM is called a separable POVM, if it can be implemented by a separable operations [16], [19], [20], [68], [69]. A POVM is separable if and only if all the elements are separable [32]: in this case, a POVM  $\{T, I - T\}$  is separable if and only if both  $T$  and  $I - T$  can be written as

$$\begin{aligned} T &= \sum_i A_i \otimes B_i \\ I - T &= \sum_i A'_i \otimes B'_i \end{aligned} \quad (1)$$

by using positive operators  $\{A_i\}_i, \{B_i\}_i, \{A'_i\}_i$ , and  $\{B'_i\}_i$ .

### B. Problem Settings

In this paper, we consider a hypothesis testing between a given fixed pure-bipartite state  $|\Psi\rangle$  and the completely mixed state (or a white noise)  $\rho_{mix}$  under the different restrictions on available POVMs: global POVM, separable POVM, one-way LOCC POVM, two-way LOCC POVM. Especially, we consider the situation where we intend to assert that an unknown state is the pure-bipartite state  $|\Psi\rangle$ . In order to do so, we choose the completely mixed state  $\rho_{mix}$  as a null hypothesis and the state  $|\Psi\rangle$  as an alternative hypothesis. That is, we minimize the error probability judging an unknown state to be  $\rho_{mix}$  when the state is actually  $|\Psi\rangle$  (the type 2 error) under the condition that a fixed value  $\alpha$  upper-bounds the error probability judging an unknown state to be  $|\Psi\rangle$  when the state is actually  $\rho_{mix}$  (the type 1 error).

Our POVM consists of two POVM elements  $T$  and  $I - T$ . When the measurement result is  $T$ , we judge an unknown state as  $|\Psi\rangle$ , and when the measurement result is  $I - T$ , we judge an unknown state as  $\rho_{mix}$ . Thus, the type 1 error is written as

$$\alpha(T) = \text{Tr} \rho_{mix} T, \quad (2)$$

and the type 2 error is written as

$$\beta(T) = \langle \Psi | (I - T) | \Psi \rangle. \quad (3)$$

As a result, the optimal type 2 error under the condition that the type 1 error is less than or equal to  $\alpha$  is written as

$$\beta_{|\Psi\rangle, C}(\alpha) \stackrel{\text{def}}{=} \min_T \{\beta(T) \mid \alpha(T) \leq \alpha, \{T, I - T\} \in C\},$$

where  $C$  is either  $\rightarrow, \leftrightarrow, Sep$ , or  $g$  corresponding to one-way LOCC, two-way LOCC, separable POVM and the global POVM, respectively. The optimal success probability  $S_{\alpha, C}(|\Psi\rangle)$  is defined as

$$S_{\alpha, C}(|\Psi\rangle) \stackrel{\text{def}}{=} 1 - \beta_{|\Psi\rangle, C}(\alpha). \quad (4)$$

In this paper, we mainly try to derive the optimal type 2 error  $\beta_{|\Psi\rangle, C}(\alpha)$  by calculating the optimal success probability  $S_{\alpha, C}(|\Psi\rangle)$ , since the latter is slightly simpler than the former.

We can easily calculate the optimal success probability for the global POVM, which apparently does not depend on choice of the pure state  $|\Psi\rangle$ . The result is

$$\beta_{|\Psi\rangle, C}(\alpha) = d_A d_B \min\{\alpha, 1/d_A d_B\}. \quad (5)$$

<sup>2</sup>We often abbreviate  $I_{AB}$  as  $I$  in the case when it is apparent on which space  $I$  is defined.

The optimal POVM is given by  $T = \beta_{|\Psi\rangle, g}(\alpha)|\Psi\rangle\langle\Psi|$ . Therefore, the purpose of this paper is evaluating  $\beta_{|\Psi\rangle, C}(\alpha)$  for  $C = \rightarrow, \leftrightarrow, Sep$ , and observing the trade-off between Type 1 error  $\alpha$  and Type 2 error  $\beta$ .

### C. Swapping null and alternative hypotheses

In this paper, we will mainly concern  $\beta_{|\Psi\rangle, C}(\alpha)$  in this paper. However, someone may be interested in the hypothesis testing whose null hypothesis and alternative hypothesis are the converses of ours. The optimal type 2 error for this converse hypothesis testing corresponds to the the optimal type 1 error  $\alpha_{|\Psi\rangle, C}(\beta)$  for our problem under the condition that type 2 error is less than a fixed value  $\beta$ :

$$\alpha_{|\Psi\rangle, C}(\beta) \stackrel{\text{def}}{=} \min_T \{ \alpha(T) \mid \beta(T) \leq \beta, \{T, I - T\} \in C \}.$$

Since the trade-off  $\beta_{|\Psi\rangle, C}(\alpha)$  is a non-decreasing function, the trade-off for the converse hypothesis testing  $\alpha_{|\Psi\rangle, C}(\beta)$  is given as

$$\alpha_{|\Psi\rangle, C}(\beta) = \min\{\alpha \mid \beta_{|\Psi\rangle, C}(\alpha) = \beta\}. \quad (6)$$

Especially, in the region where  $\beta_{|\Psi\rangle, C}(\alpha)$  is strictly decreasing, it is given just as the inverse function of  $\beta_{|\Psi\rangle, C}(\alpha)$ :

$$\alpha_{|\Psi\rangle, C}(\beta) = \beta_{|\Psi\rangle, C}^{-1}(\beta). \quad (7)$$

Actually, as we will see later,  $\beta_{|\Psi\rangle, C}(\alpha)$  is strictly decreasing all the region of  $\alpha$  except the region where  $\alpha$  satisfies  $\beta_{|\Psi\rangle, C}(\alpha) = 0$ . Therefore, the graph for the trade-off  $\alpha_{|\Psi\rangle, C}(\beta)$  is essentially derived just by swapping the axes of the graph for the trade-off  $\beta_{|\Psi\rangle, C}(\alpha)$ .

In the paper [47], we treated this converse hypothesis testing and derived the optimal type 2 error under the condition that the type 1 error is 0. In our notation, it corresponds to  $\alpha_{|\Psi\rangle, C}(0)$ . Thus, the main results of [47] can be written down as

$$\alpha_{|\Psi\rangle, Sep}(0) = \frac{1}{d_A d_B} (\text{Tr} \sqrt{\rho_A})^2, \quad (8)$$

$$\alpha_{|\Psi\rangle, \rightarrow}(0) = \frac{1}{d_A d_B} \text{rank} \rho_A, \quad (9)$$

and

$$\begin{aligned} & \alpha_{|\Psi\rangle, \leftrightarrow}(0) \\ & \leq \frac{1}{d_A d_B} \min_{\{\delta_{ki}\}_{1 \leq k \leq i \leq d}} \left\{ \sum_{i=1}^d i \cdot \frac{\sum_{k=1}^i \lambda_k \delta_{ki}^2}{\sum_{k=1}^i \lambda_k \delta_{ki}} \mid \right. \\ & \quad \left. \forall k, \forall i, \delta_{ki} \geq 0 \text{ and } \forall k, \sum_{i=k}^d \delta_{ki} = 1 \right\}, \quad (10) \end{aligned}$$

where  $d$  is defined as  $d \stackrel{\text{def}}{=} \min\{d_A, d_B\}$ , and  $\{\lambda_k\}_{k=1}^d$  is the Schmidt coefficients of  $|\Psi\rangle$  satisfying  $\lambda_k \leq \lambda_{k+1}$  for all  $k$ . Therefore, from Eq.(6), we have already known the smallest zero of  $\beta_{|\Psi\rangle, C}(\alpha)$ .

## III. MAIN RESULTS

In this section, we give the main results of this paper. In the following parts, we always choose computational basis as the Schmidt basis of  $|\Psi\rangle$  in the following way:

$$|\Psi\rangle = \sum_{i=1}^d \sqrt{\lambda_i} |ii\rangle, \quad (11)$$

where  $d \stackrel{\text{def}}{=} \min\{d_A, d_B\}$  and  $\{\lambda_i\}_{i=1}^d$  are the Schmidt coefficients of  $|\Psi\rangle$  satisfying  $\lambda_i \geq \lambda_{i+1}$ .

For one-way LOCC POVM, we prove that an optimal strategy is measuring an unknown state in each local computational basis and post-processing the measurement results. Thus, the local hypothesis testing under one-way LOCC is essentially equivalent to a classical hypothesis testing between a probability distribution defined by the Schmidt coefficients of  $|\Psi\rangle$  and the classical white noise (Lemma 2). As a result, the optimal type 2 error is given by the following theorem:

*Theorem 1:* Defining a natural number  $c$  as

$$c \stackrel{\text{def}}{=} \min\{d, \lfloor d_A d_B \alpha \rfloor + 1\}, \quad (12)$$

then, for a state  $|\Psi\rangle = \sum_i \sqrt{\lambda_i} |ii\rangle$  with  $\lambda_i \geq \lambda_{i+1}$ ,  $\beta_{|\Psi\rangle, \rightarrow}(\alpha)$  can be written as

$$\beta_{|\Psi\rangle, \rightarrow}(\alpha) = \sum_{i=c}^d \lambda_i - m_c \lambda_c, \quad (13)$$

where  $m_c$  is defined as

$$m_c \stackrel{\text{def}}{=} \min\{1, d_A d_B \alpha - c + 1\}. \quad (14)$$

An optimal POVM can be written as  $\{T, I - T\}$  by using the following  $T \in \mathfrak{B}(\mathcal{H}_{AB})$ :

$$T = \sum_{i=1}^{c-1} |ii\rangle\langle ii| + m_c |cc\rangle\langle cc|. \quad (15)$$

Since the definition of two-way LOCC is mathematically complicated in comparison to that of one-way LOCC and separable operations [16], [47], [69], it is extremely difficult to evaluate the optimal error probability for two-way LOCC POVM. Therefore, we only evaluate performance of a particular type of two-way LOCC protocols which belong to three steps LOCC and are used in the previous paper [47]. Hence, we only derive an upper bound for the optimal type 2 error for 2-way LOCC: Defining  $\tilde{\beta}_{|\Psi\rangle, \leftrightarrow}(\alpha)$  as

$$\begin{aligned} & \tilde{\beta}_{|\Psi\rangle, \leftrightarrow}(\alpha) \\ & \stackrel{\text{def}}{=} 1 - \max_{\{m_i^k\}_{1 \leq k \leq i \leq d_A}} \left\{ \sum_{i,k} \lambda_k m_i^k \mid 0 \leq m_i^k, \right. \\ & \quad \left. \sum_{i=k}^{d_A} m_i^k \leq 1, \sum_{i=1}^{d_A} i \cdot \frac{\sum_{k=1}^i \lambda_k (m_i^k)^2}{\sum_{k=1}^i \lambda_k m_i^k} \leq \alpha d_A d_B \right\}, \quad (16) \end{aligned}$$

we derive the following theorem:

*Theorem 2:*

$$\tilde{\beta}_{|\Psi\rangle, \leftrightarrow}(\alpha) \geq \beta_{|\Psi\rangle, \leftrightarrow}(\alpha). \quad (17)$$

This upper bound  $\tilde{\beta}_{|\Psi\rangle, \leftrightarrow}(\alpha)$  is in the form of a convex optimization with  $\frac{d_A(d_A+1)}{2}$  parameters.

For separable POVMs, we prove that this local hypothesis testing problem is equivalent to another hypothesis testing problem with a composite null hypothesis under global POVM, and by solving this simpler hypothesis testing problem, we derive an optimal type 2 error for the original local hypothesis testing problem. Here, we only give the final theorem for the local hypothesis testing under separable POVM. First, we can assume  $d_A \leq d_B$  without losing generality. For given  $\alpha > 0$  and  $|\Psi\rangle$ , we introduce the following notations: For a natural number  $l \leq d_A$ , a real number  $\epsilon_l$  is defined as  $\epsilon_l \stackrel{\text{def}}{=} \sqrt{\frac{\alpha d_A d_B}{l}}$ , a state  $|\psi_l\rangle$  is defined as

$$|\psi_l\rangle \stackrel{\text{def}}{=} \sum_{i=1}^l \sqrt{\lambda_i} |i\rangle / \sqrt{\sum_{i=1}^l \lambda_i}, \quad (18)$$

a state  $|\phi_l\rangle$  is defined as

$$|\phi_l\rangle = \frac{1}{\sqrt{l}} \sum_{i=1}^l |i\rangle, \quad (19)$$

and a state  $|\phi'_l\rangle$  is defined as

$$|\phi'_l\rangle = \frac{\sqrt{1-\epsilon_l^2}|\psi_l\rangle - (c_l\sqrt{1-\epsilon_l^2} - \epsilon_l\sqrt{1-c_l^2})|\phi_l\rangle}{\sqrt{1-c_l^2}}, \quad (20)$$

where  $c_l$  is defined as  $c_l \stackrel{\text{def}}{=} \langle \psi_l | \phi_l \rangle$ . By using the above notations, a natural number  $\eta$  is defined as

$$\eta \stackrel{\text{def}}{=} \begin{cases} d_A & \text{if } \epsilon_{d_A} \geq \langle \phi_{d_A} | \psi_{d_A} \rangle \text{ or } |\psi_{d_A}\rangle = |\phi_{d_A}\rangle \\ \text{otherwise} & \\ \min_{l \in \mathbb{N}} \left\{ l \mid \begin{array}{l} l \leq d, \epsilon_l < \langle \phi_l | \psi_l \rangle, \\ |\psi_l\rangle \neq |\phi_l\rangle, \langle l | \phi'_l \rangle < 0 \end{array} \right\} - 1 & \end{cases} \quad (21)$$

By the definition,  $\eta$  satisfies  $1 \leq \eta \leq d_A$ . Further, we define an operator  $T(|\phi\rangle)$  depending on a vector  $|\phi\rangle \in \mathcal{H}_A$  as

$$T(|\phi\rangle) \stackrel{\text{def}}{=} V|\phi\rangle\langle\phi|V^\dagger + \sum_{j \neq k} \sqrt{\langle j | \phi \rangle \langle \phi | k \rangle} |j\rangle\langle j| \otimes |k\rangle\langle k|.$$

In the above formula,  $V$  is an isometry between  $\mathcal{H}_A$  and  $\mathcal{H}_{AB}$  defined as  $V \stackrel{\text{def}}{=} \sum_i |ii\rangle\langle i|$ . As we will prove later,  $\{T(|\phi\rangle), I - T(|\phi\rangle)\}$  is a separable POVM for all  $|\phi\rangle \in \mathcal{H}_A$ . Then, by using the above notations, the optimal type 2 error is given by the following theorem:

**Theorem 3:** 1) In the case when  $\epsilon_\eta \geq \langle \phi_\eta | \psi_\eta \rangle$ ,

$$\beta_{|\Psi\rangle, \text{Sep}}(\alpha) = 1 - \sum_{i=1}^{\eta} \lambda_i, \quad (22)$$

and a POVM  $T(|\psi_\eta\rangle)$  attains the optimum.

2) In the case when  $\epsilon_\eta < \langle \phi_\eta | \psi_\eta \rangle$ ,

$$\beta_{|\Psi\rangle, \text{Sep}}(\alpha) = 1 - \left( \sum_{i=1}^{\eta} \lambda_i \right) \cdot \left( \sqrt{1-\epsilon_\eta^2} \sqrt{1-c_\eta^2} + \epsilon_\eta c_\eta \right)^2 \quad (23)$$

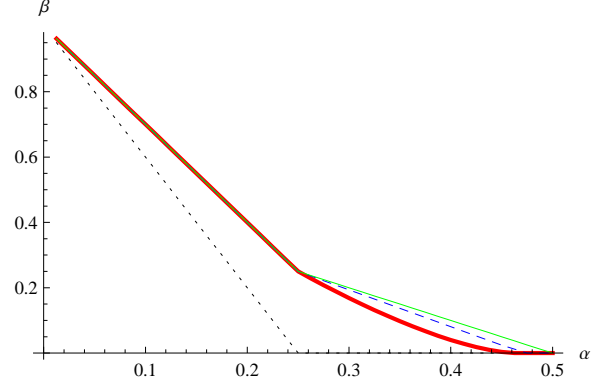


Fig. 1. The trade-off between the type 1 error ( $\alpha$ ) and the type 2 error ( $\beta$ ) for  $|\Psi\rangle = \frac{\sqrt{3}}{2}|11\rangle + \frac{1}{2}|22\rangle$ . Thin line:  $\beta_{|\Psi\rangle, \rightarrow}(\alpha)$ ; Broken line:  $\tilde{\beta}_{|\Psi\rangle, \leftrightarrow}(\alpha)$ ; Thick line:  $\beta_{|\Psi\rangle, \text{Sep}}(\alpha)$ ; Dotted line:  $\beta_{|\Psi\rangle, g}(\alpha)$ .

A POVM  $T(|\phi'_\eta\rangle)$  attains the optimum in the case  $|\psi_\eta\rangle \neq |\phi_\eta\rangle$ , a POVM  $T(\epsilon_1|1\rangle)$  attains the optimum in the case  $\eta = 1$ , and a POVM  $T(\epsilon_\eta|\phi_\eta\rangle + \sqrt{1-\epsilon_\eta^2}|\phi_\eta^\perp\rangle)$  attains the optimum in the case  $\eta \geq 2$  and  $|\psi_\eta\rangle = |\phi_\eta\rangle$ . Here,  $|\phi_\eta^\perp\rangle$  is any state orthogonal to  $|\phi_\eta\rangle$  and, thus, can be chosen as  $|\phi_\eta^\perp\rangle = (|1\rangle - |2\rangle)/\sqrt{2}$ .

Before discussing plots of  $\beta_{|\Psi\rangle, C}(\alpha)$ , we explain several facts which can be easily seen from the above main theorems. For the global POVM, we can trivially derive  $\beta_{|\Psi\rangle, g}(\alpha) = 0$  for  $\alpha \geq 1/d_A d_B$ . On the other hand, for the other local POVMs, we derive  $\beta_{|\Psi\rangle, \text{Sep}}(\alpha) = \beta_{|\Psi\rangle, \leftrightarrow}(\alpha) = \beta_{|\Psi\rangle, \rightarrow}(\alpha) = 0$  for  $\alpha \leq 1/\max\{d_A, d_B\}$ . The latter can be easily seen from Theorem 1. Moreover, we can derive the following corollary from the above theorem:

**Corollary 1:** For  $\alpha < 1/d_A d_B$ ,

$$\beta_{|\Psi\rangle, \text{Sep}}(\alpha) = \beta_{|\Psi\rangle, \leftrightarrow}(\alpha) = \beta_{|\Psi\rangle, \rightarrow}(\alpha) = 1 - \lambda_1 \alpha d_A d_B. \quad (24)$$

The optimal POVM is given by  $T = 1 - \lambda_1 \alpha d_A d_B$ . When  $|\Psi\rangle$  is a product state or a maximally entangled state, Eq.(24) holds all the region  $0 \leq \alpha \leq 1/\max\{d_A, d_B\}$

*Proof:* See Appendix A.

Thus, separable and one-way and two-way LOCC POVM just give the same optimal error for  $\alpha < 1/d_A d_B$  and  $\alpha > 1/\max\{d_A, d_B\}$  for a non-maximally entangled state  $|\Psi\rangle$ . On the other hand, they just coincide in all the region for a maximally entangled state  $|\Psi\rangle$ .

Now, we present several figures about graphs of the trade-off between the type 1 error  $\alpha$  and the type 2 error  $\beta$  for global, separable, two-way LOCC, and one-way LOCC POVM. For two-way LOCC POVM, we draw the graph of  $\tilde{\beta}_{|\Psi\rangle, \leftrightarrow}(\alpha)$  instead of  $\beta_{|\Psi\rangle, \leftrightarrow}(\alpha)$ . Here, we always choose  $d_A = d_B = d$  for simplicity. First, we give graphs of the trade-off for  $|\Psi\rangle = \frac{\sqrt{3}}{2}|11\rangle + \frac{1}{2}|22\rangle$  (FIG. 1) and  $|\Psi\rangle = \left(\frac{\sqrt{3}}{2}|11\rangle + \frac{1}{2}|22\rangle\right)^{\otimes 4}$  (FIG. 2). The graphs for separable, one-way LOCC and two-way LOCC coincide in the regions  $\alpha \leq 1/d^2$  and  $\alpha \leq 1/d$ . On the other hands, they separate in all the region  $1/d^2 < \alpha < 1/d$ , that is,  $\beta_{|\Psi\rangle, \text{Sep}}(\alpha)$  is strictly smaller than  $\tilde{\beta}_{|\Psi\rangle, \leftrightarrow}(\alpha)$ , and also  $\tilde{\beta}_{|\Psi\rangle, \leftrightarrow}(\alpha)$  is smaller than  $\beta_{|\Psi\rangle, \rightarrow}(\alpha)$ . In the previous paper [47], we observed improvement of the

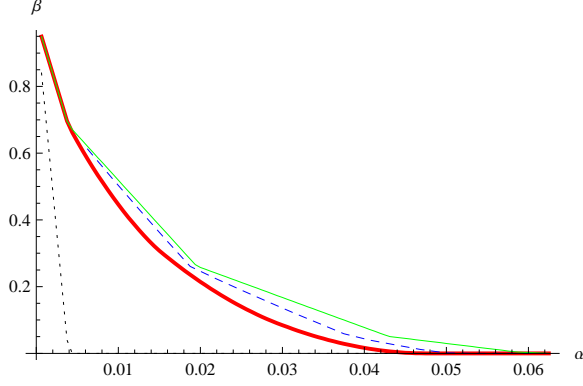


Fig. 2. The trade-off between the type 1 error ( $\alpha$ ) and the type 2 error ( $\beta$ ) for  $|\Psi\rangle = \{\frac{\sqrt{3}}{2}|11\rangle + \frac{1}{2}|22\rangle\}^{\otimes 4}$ . Thin line:  $\beta_{|\Psi\rangle, \rightarrow}(\alpha)$ ; Broken line:  $\tilde{\beta}_{|\Psi\rangle, \leftrightarrow}(\alpha)$ ; Thick line:  $\beta_{|\Psi\rangle, sep}(\alpha)$ ; Dotted line:  $\beta_{|\Psi\rangle, g}(\alpha)$ .

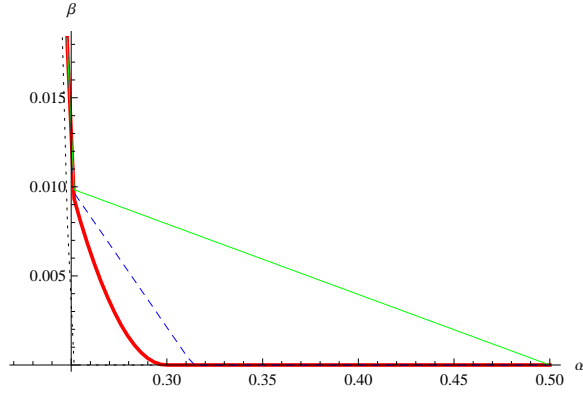


Fig. 3. The trade-off between the type 1 error ( $\alpha$ ) and the type 2 error ( $\beta$ ) for  $|\Psi\rangle = \frac{10}{\sqrt{101}}|11\rangle + \frac{1}{\sqrt{101}}|22\rangle$ . Thin line:  $\beta_{|\Psi\rangle, \rightarrow}(\alpha)$ ; Broken line:  $\tilde{\beta}_{|\Psi\rangle, \leftrightarrow}(\alpha)$ ; Thick line:  $\beta_{|\Psi\rangle, sep}(\alpha)$ ; Dotted line:  $\beta_{|\Psi\rangle, g}(\alpha)$ .

optimal error probability from one-way (two-steps) LOCC to three-steps two-way LOCC by using the same simple three-steps LOCC protocol used in this paper for  $\tilde{\beta}_{|\Psi\rangle, \leftrightarrow}(\alpha)$ . As we have explained, these optimal error probabilities in the previous paper correspond to the smallest zeros of the graphs  $\tilde{\beta}_{|\Psi\rangle, \leftrightarrow}(\alpha)$  and  $\beta_{|\Psi\rangle, \rightarrow}(\alpha)$  in the present paper. The presented graphs show that the similar improvement is observed all the region of  $1/d^2 < \alpha < 1/d$ . As we can observe from FIG 3, when  $|\Psi\rangle$  just have small entanglement, this improvement can be seen more clearly. In other words, in this case, the straight line  $\tilde{\beta}_{|\Psi\rangle, \leftrightarrow}(\alpha)$  gives an approximation of the curve  $\beta_{|\Psi\rangle, sep}(\alpha)$  in the region  $1/d^2 < \alpha < 1/d$ . Finally, we give a graph showing the variation with  $|\Psi\rangle$  of  $\beta_{|\Psi\rangle, C}(\alpha)$  for a fixed  $\alpha$  (FIG 4). As we have explained in Corollary 1, the graphs coincide when  $|\Psi\rangle$  is a product state ( $\lambda = 0$ ) and a maximally entangled state ( $\lambda = 0.5$ ). On the other hand, the difference between  $\beta_{|\Psi\rangle, \rightarrow}(\alpha)$  and  $\tilde{\beta}_{|\Psi\rangle, \leftrightarrow}(\alpha)$  is maximized when  $\beta$  is closed to 0.

#### IV. HYPOTHESIS TESTING UNDER LOCC

In this section, we treat the hypothesis testing under LOCC. In the first subsection, we treat one-way LOCC and give a proof of Theorem 1. In the second subsection, we give a detail

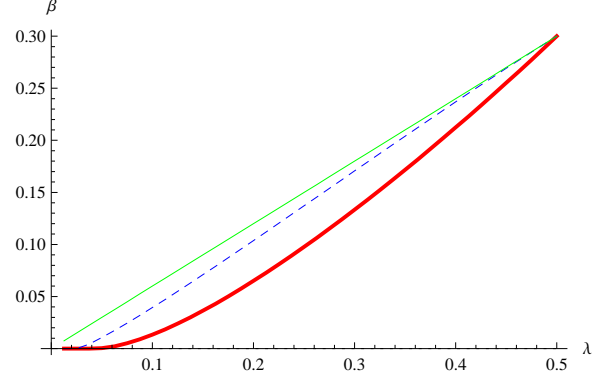


Fig. 4. Type 2 error as a function of  $\lambda$  with  $\alpha = 0.35$ , where  $\lambda$  is defined as  $|\Psi\rangle = \sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle$ . Thin line:  $\beta_{|\Psi\rangle, \rightarrow}(\alpha)$ ; Broken line:  $\tilde{\beta}_{|\Psi\rangle, \leftrightarrow}(\alpha)$ ; Thick line:  $\beta_{|\Psi\rangle, sep}(\alpha)$ .

discussion about two-way LOCC protocols including a proof of Theorem 2.

##### A. One-way LOCC

The main purpose of this subsection is giving a proof of Theorem 1, which gives an optimal type 2 error probability  $\beta_{|\Psi\rangle, \rightarrow}(\alpha)$  under one-way LOCC.

When we consider one-way LOCC on a bipartite system [47], [69], [16], there are two possibilities for a direction of classical communications, that is, from Alice  $\mathcal{H}_A$  to Bob  $\mathcal{H}_B$  and from Bob  $\mathcal{H}_B$  to Alice  $\mathcal{H}_A$ . Here, since a state  $|\Psi\rangle$  is symmetric under the swapping between Alice and Bob, we can restrict ourselves into the situation where Alice send a classical message to Bob without losing generality. Thus, we are interested in an optimal success probability  $S_{\alpha, \rightarrow}$  defined as

$$S_{\alpha, \rightarrow}(|\Psi\rangle) \stackrel{\text{def}}{=} \max_T \{ \langle \Psi | T | \Psi \rangle | \text{Tr} \rho_{mix} T \leq \alpha, \{T, I - T\} \in \rightarrow \}, \quad (25)$$

where  $\rightarrow$  is a set of all one-way LOCC POVMs.

We first derive the following lemma, which reduces our local hypothesis testing problem to a hypothesis testing problem defined just on a single Hilbert space:

*Lemma 1:*

$$S_{\alpha, \rightarrow}(|\Psi\rangle) = \max_{M \in \mathcal{B}(\mathcal{H}_A)} \left\{ \text{Tr} \rho_A M \mid \text{Tr} M \leq d_A d_B \alpha, 0 \leq M \leq I_A \right\}, \quad (26)$$

where  $\rho_A$  is a reduced density matrix of  $|\Psi\rangle$ ;  $\rho_A \stackrel{\text{def}}{=} \text{Tr}_B |\Psi\rangle \langle \Psi|$ .

*Proof:* Without losing generality, we can choose Alice's POVM as a rank one POVM. Thus, an optimal POVM can be written as  $T = \sum_m |m\rangle \langle m| \otimes N_m$ , where  $\sum_m |m\rangle \langle m| = I_A$  and  $0 \leq N_m \leq I_B$ . After Alice's measurement, Bob's system  $\mathcal{H}_B$  is in a state  $|p_m\rangle \stackrel{\text{def}}{=} \langle m | \Psi \rangle / \|\langle m | \Psi \rangle\|$ . Suppose  $T'$  is defined as

$$T' \stackrel{\text{def}}{=} \sum_m \langle p_m | N_m | p_m \rangle |m\rangle \langle m| \otimes |p_m\rangle \langle p_m|. \quad (27)$$

Then, this new one-way LOCC POVM  $T'$  satisfies  $\langle \Psi | T' | \Psi \rangle = \langle \Psi | T | \Psi \rangle$  and  $\text{Tr} T' \leq \text{Tr} T$ . Thus,  $T'$  is also an optimal POVM. Defining  $M \stackrel{\text{def}}{=} \sum_m \| |m\rangle \|^2 \cdot \langle p_m | N_m | p_m \rangle$ , we have  $\text{Tr} T' = \text{Tr} M$ . Moreover,  $\langle \Psi | T' | \Psi \rangle$  can be evaluated as

$$\begin{aligned} \langle \Psi | T' | \Psi \rangle &= \sum_m \| |m\rangle \|^2 \cdot \langle p_m | N_m | p_m \rangle \\ &= \text{Tr} \rho_A M. \end{aligned} \quad (28)$$

$0 \leq \langle p_m | N_m | p_m \rangle \leq 1$  and  $\sum_m |m\rangle \langle m| = I_A$  guarantees  $0 \leq M \leq I_A$ . Therefore, we derive

$$\begin{aligned} S_{\alpha, \rightarrow}(|\Psi\rangle) &\leq \max_{M \in \mathcal{B}(\mathcal{H}_A)} \left\{ \text{Tr} \rho_A M \mid \text{Tr} M \leq d_A d_B \alpha, 0 \leq M \leq I_A \right\}. \end{aligned} \quad (29)$$

On the other hand, suppose an operator  $M$  attains the optimum of the right-hand side of the above inequality, and has a spectral decomposition  $M = \sum_m q_m |m\rangle \langle m|$ . By defining a one-way LOCC POVM element  $T$  as  $T \stackrel{\text{def}}{=} \sum_m q_m |m\rangle \langle m| \otimes |p_m\rangle \langle p_m|$ , where  $|p_m\rangle \stackrel{\text{def}}{=} \langle m | \Psi \rangle / \| \langle m | \Psi \rangle \|$ , we can easily see that this POVM element attains Eq.(29). Therefore, we derive Eq.(26).  $\square$

We further reduce  $S_{\alpha, \rightarrow}(|\Psi\rangle)$  as follows:

*Lemma 2:*

$$\begin{aligned} S_{\alpha, \rightarrow}(|\Psi\rangle) &= \max \left\{ \sum_{i=1}^r \lambda_i m_i \mid \sum_{i=1}^{d_A} m_i \leq d_A d_B \alpha, 0 \leq m_i \leq 1 \right\}. \end{aligned} \quad (30)$$

*Proof:* By the definition,  $\rho_A$  can be written as  $\rho_A = \sum_i \lambda_i |i\rangle \langle i|$ . Suppose  $M$  is optimal. Then, we can define a new operator  $M'$  by means of pinching as  $M' = \sum_i \langle i | M | i \rangle |i\rangle \langle i|$ . It is straightforward to check  $\text{Tr} M' = \text{Tr} M$ ,  $\text{Tr} \rho_A M' = \text{Tr} \rho_A M$ , and  $0 \leq M' \leq I_A$ . Thus,  $M'$  is also optimal. Hence, we can always choose an optimal  $M$  as  $M = \sum_i m_i |i\rangle \langle i|$ . Thus, we derive Eq.(30).  $\square$

This lemma shows that the local hypothesis testing under one-way LOCC is essentially equivalent to a hypothesis testing of two classical probability distributions:  $\{\lambda_i\}_{i=1}^{d_A}$  and  $\{1/d_A\}_{i=1}^{d_A}$ .

By means of the above lemma, we can give a proof of Theorem 1, which gives an analytical solution for the hypothesis testing under one-way LOCC as follows:

*Proof (Theorem 1):* From the above lemma, we can choose  $m_i = 0$  for all  $i > r$ . In the case when  $r \leq d_A d_B \alpha$ , an optimum is attained when  $m_i = 1$  for all  $0 \leq i \leq r$ , and we have  $S_{\alpha, \rightarrow}(|\Psi\rangle) = 1$ . Thus, we only consider the case when  $r > d_A d_B \alpha$  in the following part.

Suppose  $\{m_i\}_{i=1}^r$  attains the optimum. First, we prove  $\sum_{i=1}^r m_i = d_A d_B \alpha$  by contradiction. Suppose  $\sum_{i=1}^r m_i < d_A d_B \alpha$ . Then, there exists  $i_0$  such that  $m_{i_0} < 1$ . Thus, there exists  $\epsilon > 0$  such that  $m_{i_0} + \epsilon \leq 1$ . By defining  $m'_{i_0} = m_{i_0} + \epsilon$  and  $m'_i = m_i$  for all  $i \neq i_0$ ,  $\{m'_i\}_{i=1}^r$  satisfies  $\sum_{i=1}^r \lambda_i m'_i > \sum_{i=1}^r \lambda_i m_i$ . Thus,  $\{m_i\}_{i=1}^r$  is not optimal. This is contradiction. Therefore,  $\sum_{i=1}^r m_i = d_A d_B \alpha$ .

Second, we prove that an optimal  $\{m_i\}_{i=1}^r$  satisfies  $m_i = 1$  for all  $i \leq c - 1$  and  $m_i = 0$  for all  $i > c$  by contradiction.

For an optimal  $\{m_i\}_{i=1}^r$ , suppose there exists a pair of natural numbers  $k$  and  $l$  such that  $k < l \leq r$ ,  $m_k < 1$  and  $m_l > 0$ . Then, by defining  $\{m'_i\}_{i=1}^r$  as  $m'_k \stackrel{\text{def}}{=} \min\{1, m_k + m_l\}$ ,  $m'_l \stackrel{\text{def}}{=} \max\{0, m_l - (1 - m_k)\}$ , and  $m'_i = m_i$  for all  $i$  satisfying  $i \neq k$  and  $i \neq l$ , we derive  $\sum_{i=1}^r m'_i = d_A d_B \alpha$ . We have  $\sum_{i=1}^r \lambda_i m'_i > \sum_{i=1}^r \lambda_i m_i$  for  $\lambda_k > \lambda_l$ , and  $\sum_{i=1}^r \lambda_i m'_i = \sum_{i=1}^r \lambda_i m_i$  for  $\lambda_k = \lambda_l$ . Thus, when  $\lambda_k > \lambda_l$ , this is contradiction, and when  $\lambda_k = \lambda_l$ ,  $\{m'_i\}_{i=1}^r$  also gives an optimal POVM. Thus, when  $k < l \leq r$  and  $\lambda_k > \lambda_l$ , we have either  $m_k = 1$  or  $m_l = 0$ . Therefore, there exist  $c_1$  and  $c_2$  such that an optimal  $\{m_i\}_{i=1}^r$  satisfies  $m_i = 1$  for all  $i < c_1$ ,  $m_i = 0$  for all  $i \geq c_2$ , and  $\lambda_{c_1} = \dots = \lambda_{c_2-1}$ . Thus, suppose  $\{m_i\}_{i=1}^r$  is optimal.  $\{m'_i\}_{i=1}^r$  is also optimal when it satisfies  $m_i = 1$  for all  $i < c_1$ ,  $m_i = 0$  for all  $i \geq c_2$ , and  $\sum_{i=c_1}^{c_2-1} m'_i = \sum_{i=c_1}^{c_2-1} m_i$ . Especially, we can choose an optimal  $\{m_i\}_{i=1}^r$  as one satisfying  $m_i = 1$  for all  $i < c$ ,  $m_i = 0$  for all  $i \geq c + 1$  for  $c$  defined by

$$c \stackrel{\text{def}}{=} \max_{n \in \mathbb{Z}_+} \{n \mid n \leq d_A d_B \alpha\} + 1. \quad (31)$$

In this case  $m_c$  can be written down as

$$m_c \stackrel{\text{def}}{=} d_A d_B \alpha - c + 1. \quad (32)$$

$\square$

Finally, Theorem 1, the optimal 1-way LOCC strategy can be described as follows: Alice and Bob independently measure their system in the Schmidt basis. When they get the measurement result  $|ii\rangle$  for  $i \leq c - 1$ , they judge the given state to be  $|\Psi\rangle$ . When they get  $|cc\rangle$ , they conclude  $|\Psi\rangle$  in the probability  $1 - m_c$ , and in all other cases, they judge the state to be  $\rho_{\text{mix}}$ .

## B. Two-way LOCC

In this subsection, we treat the hypothesis testing under the restriction of two-way LOCC. The definition of two-way LOCC is mathematically complicated in comparison to that of one-way LOCC and separable operations [16], [47], [69]. Hence, it is extremely difficult to evaluate optimal performance of information processing under the restriction of two-way LOCC except in the case when we only concern the first exponent of asymptotics (see Section 3.5 of [16]), or when we can prove the optimal performance with two-way LOCC is the same as that with one-way LOCC, or separable operations. Therefore, we only evaluate performance of a particular type of two-way LOCC protocols belonging to three steps LOCC by a numerical optimization.

Suppose a bipartite state  $|\Psi\rangle \in \mathcal{H}_{AB}$  is shared by Alice ( $\mathcal{H}_A$ ) and Bob ( $\mathcal{H}_B$ ). Then, without losing generality, we can assume that a given three-steps protocol consists of the first Alice's measurement  $\{M_i\}_{i \in I}$ , the first Bob's measurement  $\{N_j^i\}_{j \in J}$  depending on the first Alice's measurement results  $i$ , and the second Alice's measurement  $\{L^{ij}, I_A - L^{ij}\}$  depending on the first Alice and Bob's measurement results  $i$  and  $j$ . If the first Alice and Bob's measurement results satisfy  $i \in I_0 \subset I$  and  $j \in J_0 \subset J$ , and Alice gets  $L^{ij}$  as the second measurement result, she judges that the given state is  $|\Psi\rangle$ , and

otherwise, she judges that it is  $\rho_{mix}$ . Thus, we can write down a POVM element corresponding to  $|\Psi\rangle$  as

$$T = \sum_{i \in I_0, j \in J_0} \sqrt{M_i} L^{ij} \sqrt{M_i} \otimes N_j^j, \quad (33)$$

where  $0 \leq \sum_i M_i \leq I_A$ ,  $0 \leq \sum_j N_j^j \leq I_B$ , and  $0 \leq L^{ij} \leq I_A$ . Without losing generality, we can assume that Bob never judges whether a given state is  $|\Psi\rangle$  or  $\rho_{mix}$ ; that is, Alice makes all decisions. Then, since Bob's state after Alice's first measurement can be written down as  $\{(\sqrt{\rho_A} M_i \sqrt{\rho_A}) > 0\}$ , an optimal Bob's measurement can satisfy  $\sum_{j \in J_0} N_j^j = \{(\sqrt{\rho_A} M_i \sqrt{\rho_A}) > 0\}$ , where  $\{X > 0\}$  is an orthogonal projection to the subspace spanned by all eigen vectors of  $X$  corresponding to strictly positive eigen values [16].

An optimal success probability  $\langle \Psi | T | \Psi \rangle$  under the above restrictions is still too complicated to get the value by numeric. Even in the case  $d_A = d_B = 2$ , the optimization problem is non-convex nonlinear programming including unlimited number of parameters. Thus, here, we only consider a particular type of protocols which are derived from the three step LOCC protocol used in [47] by small modifications. The protocol is derived by means of the following two restriction from general three step LOCC protocols.

- 1) As a first assumption, we choose  $|\xi_j^i\rangle\langle\xi_j^i|$  as Bob's measurement  $N_j^i$ , where  $\{|\xi_j^i\rangle\}_{j=1}^{\text{rank } M_i}$  is a mutually unbiased basis for the eigen basis of Bob's state after Alice's first measurement [4]. It is known that Bob can send all the quantum information of his system to Alice by measurements in a mutually unbiased basis when Alice and Bob's system is in a pure state [23], [41]. Since when a given state is  $|\Psi\rangle$ , the state after Alice's measurement is a pure state, Bob can send all the information for Alice in this case.
- 2) Second, we assume that in the final step, Alice's detect  $\sigma_A^{ij}$  in probability one, where

$$\sigma_A^{ij} \stackrel{\text{def}}{=} \frac{\sqrt{M_i} \rho_A N_j^{iT} \sqrt{\rho_A} M_i}{\text{Tr}(\sqrt{\rho_A} M_i \sqrt{\rho_A} N_j^{iT})} \quad (34)$$

is Alice's state after Bob's measurement when a given state is  $|\Psi\rangle$ . Hence,  $L^{ij}$  can be written down as  $L^{ij} = \{\sigma_A^{ij} > 0\}$ .

We define  $\bar{S}_{\alpha, \leftrightarrow}(|\Psi\rangle)$  as the optimal success probability under these two assumptions:

$$\begin{aligned} & \bar{S}_{\alpha, \leftrightarrow}(|\Psi\rangle) \\ & \stackrel{\text{def}}{=} \max_T \left\{ \langle \Psi | T | \Psi \rangle \mid \text{Tr} T \leq \alpha d_A d_B, \right. \\ & T = \sum_{i \in I_0} \sum_{j=1}^{\text{rank } M_i} \sqrt{M_i} \{\sigma_A^{ij} > 0\} \sqrt{M_i} \otimes |\xi_j^i\rangle\langle\xi_j^i|, \\ & \left. 0 \leq \sum_{i \in I_0} M_i \leq I_A \right\}. \end{aligned} \quad (35)$$

Then,  $\bar{S}_{\alpha, \leftrightarrow}(|\Psi\rangle)$  satisfies:

*Lemma 3:*

$$S_{\alpha, \rightarrow}(|\Psi\rangle) \leq \bar{S}_{\alpha, \leftrightarrow}(|\Psi\rangle) \leq S_{\alpha, \leftarrow}(|\Psi\rangle) \quad (36)$$

*Proof:* The second inequality is trivial from the definition of  $\bar{S}_{\alpha, \leftrightarrow}(|\Psi\rangle)$ . In order to see the first inequality, we need to choose  $I_0 = \{1, \dots, c\}$ ,  $M_i = |i\rangle\langle i|$  for  $1 \leq i \leq c-1$ , and  $M_c = m_c |c\rangle\langle c|$  in Eq. (35), where  $c$  and  $m_c$  are defined by Eq.(12) and Eq.(14). Then,  $T$  defined in Eq. (35) coincides the optimal one-way LOCC POVM given in Eq.(15)  $\square$

The optimization of  $\bar{S}_{\alpha, \leftrightarrow}(|\Psi\rangle)$  can be reduced as follows:

*Lemma 4:*

$$\begin{aligned} & \bar{S}_{\alpha, \leftrightarrow}(|\Psi\rangle) \\ & = \max_{\{m_i^k\}_{i \in \mathcal{P}(d_A), k \in i}} \left\{ \sum_{i, k} \lambda_k m_i^k \mid 0 \leq m_i^k, \sum_{i \in \mathcal{P}(d_A)} m_i^k \leq 1, \right. \\ & \left. \sum_{i \in \mathcal{P}(d_A)} |i| \cdot \frac{\sum_{k \in i} \lambda_k (m_i^k)^2}{\sum_{k \in i} \lambda_k m_i^k} \leq \alpha d_A d_B \right\}, \end{aligned} \quad (37)$$

where  $\mathcal{P}(d_A)$  is a power set (a set of all subsets) of  $\{1, \dots, d_A\}$ ,  $|i|$  is a number of elements in the set  $i$ , and  $\rho_A = \sum_{k=1}^{d_A} \lambda_k |k\rangle\langle k|$ .

*Proof:* First, by straightforward calculations, we derive

$$\begin{aligned} \langle \Psi | T | \Psi \rangle & = \sum_i \rho_A M_i, \\ \text{Tr} T & = \sum_i \text{rank} M_i \frac{\text{Tr} \rho_A M_i^2}{\text{Tr} \rho_A M_i}. \end{aligned}$$

By using a pinching technique [16] in the eigen basis of  $\rho_A$ , we observe that  $M_i$  can be chosen as to be a diagonal matrix in this basis. Moreover, we only need to consider POVM  $\{M_i\}_{i \in I_0}$  in which support of  $M_i$  is different from  $M_j$  for all  $i \neq j$ . This can be shown as follows: Suppose  $M_i$  and  $M_j$  have the same support for an optimal  $\{M_i\}_{i \in I_0}$ . We define a new POVM  $T'$  by using  $\{M'_i\}_{i \in I_0}$  which is defined as  $M'_i = M_i + M_j$ ,  $M'_j = 0$  and  $M'_k = M_k$  for all  $k \neq i, j$ . Then, we have

$$\begin{aligned} & (\text{Tr} T - \text{Tr} T') / \text{rank} M_i \\ & = \left( \langle M_i \rangle^2 \langle M_j^2 \rangle - 2 \langle M_i M_j \rangle \langle M_i \rangle \langle M_j \rangle \right. \\ & \quad \left. + \langle M_i^2 \rangle \langle M_j \rangle^2 \right) / (\langle M_i \rangle \langle M_j \rangle \langle M_i + M_j \rangle) \\ & \geq \frac{\left( \sqrt{\langle M_i^2 \rangle} \langle M_j \rangle - \sqrt{\langle M_j^2 \rangle} \langle M_i \rangle \right)^2}{\langle M_i \rangle \langle M_j \rangle \langle M_i + M_j \rangle} \\ & \geq 0, \end{aligned} \quad (38)$$

where  $\langle M \rangle$  is abbreviation of  $\text{Tr} \rho_A M$ , and we used the Schwarz inequality in the first inequality. Thus, we have  $\text{Tr} T' \leq \text{Tr} T$ , and  $T'$  is also optimal when  $T$  is optimal. Thus, we can choose  $\mathcal{P}(d_A)$  as  $I_0$ . Finally, by just defining  $m_i^k$  as  $M_i = \sum_{k \in i} m_i^k |k\rangle\langle k|$ , we derive Eq.(37).  $\square$

By direct calculation, we can check the function  $\sum_{i \in \mathcal{P}(d_A)} |i| \cdot \frac{\sum_{k \in i} \lambda_k m_i^k}{\sum_{k \in i} \lambda_k m_i^k}$  is a convex function. Therefore, the optimization problem in Eq.(37) is a convex optimization. Thus, its local optimum is the global optimum, and we can easily access the optimum by numerics at least for a small dimensional system.

Up to now, we have presented a mathematically rigorous reduction of  $\bar{S}_{\alpha, \leftrightarrow}(|\Psi\rangle)$  and derived Eq.(37). On the other hand, although we do not have any proof, numerical

calculations strongly suggest that  $\bar{S}_{\alpha, \leftrightarrow}(|\Psi\rangle)$  further can be reduced in the following way: By adding further restrictions onto Eq.(37) as  $M_i = 0$  for all  $i \in \mathcal{P}(d_A)$  except  $i = \{1\}, \{1, 2\}, \dots, \{1, \dots, d_A\}$ , we define  $\tilde{S}_{\alpha, \leftrightarrow}(|\Psi\rangle)$  as

$$\begin{aligned} & \tilde{S}_{\alpha, \leftrightarrow}(|\Psi\rangle) \\ &= \max_{\{m_i^k\}_{1 \leq k \leq i \leq d_A}} \left\{ \sum_{i,k} \lambda_k m_i^k \mid 0 \leq m_i^k, \sum_{i=k}^{d_A} m_i^k \leq 1, \right. \\ & \quad \left. \sum_{i=1}^{d_A} i \cdot \frac{\sum_{k=1}^i \lambda_k (m_i^k)^2}{\sum_{k=1}^i \lambda_k m_i^k} \leq \alpha d_A d_B \right\}, \end{aligned} \quad (39)$$

This optimization problem is a convex optimization with just  $O(d_A^2)$  parameters. Our numerical calculations strongly suggest  $\tilde{S}_{\alpha, \leftrightarrow}(|\Psi\rangle) = \bar{S}_{\alpha, \leftrightarrow}(|\Psi\rangle)$ . Even if this equality is not true, we trivially have  $\tilde{S}_{\alpha, \leftrightarrow}(|\Psi\rangle) \leq \bar{S}_{\alpha, \leftrightarrow}(|\Psi\rangle)$ . Thus,  $\tilde{S}_{\alpha, \leftrightarrow}(|\Psi\rangle)$  is also a lower bound of  $S_{\alpha, \leftrightarrow}(|\Psi\rangle)$ . We can define the optimal type 2 error under the three assumptions as  $\tilde{\beta}_{|\Psi, \leftrightarrow}(\alpha) \stackrel{\text{def}}{=} 1 - \tilde{S}_{\alpha, \leftrightarrow}(|\Psi\rangle)$ . Then, we have  $\tilde{\beta}_{|\Psi, \leftrightarrow}(\alpha) \geq \beta_{|\Psi, \leftrightarrow}(\alpha)$ . This completes the proof of Theorem 2.

## V. GLOBAL HYPOTHESIS TESTING WITH A COMPOSITE ALTERNATIVE HYPOTHESIS

As a preparation for the next section, we treat a global hypothesis testing having a composite alternative hypothesis in this section. As we will prove in the next section, this relatively simpler hypothesis testing is actually equivalent to the local hypothesis testing under separable POVM. The organization of the section is as follows: We explain the problem settings and the relation between the global hypothesis testing and the local hypothesis testing under separable POVM in the subsection A. Then, we reduce the global hypothesis testing with a composite alternative hypothesis to a hypothesis testing with a simple alternative hypothesis with an additional restriction on POVM in the subsection B. Finally, we derive analytical solutions for the problem in the subsection C.

### A. Preliminary for the section

In the conventional (classical) statistical inference, a hypothesis testing normally has a composite hypothesis (a hypothesis consists of a set of probability distributions) in practical situations, and a hypothesis testing with simple null and alternative hypotheses is usually treated in pure theoretical motivation, like the Neyman-Pearson lemma, Stein's lemma, and Chernoff bound. On the other hand, in quantum statistical inference, so far, just a very limited number of works treat a hypothesis testing with a composite hypothesis [55], [57], [67]. In this section, we add one example into this category. Our example consists of a simple alternative hypothesis and a composite null hypothesis on a single partite Hilbert-space  $\mathcal{H}$ : A null hypothesis is a composite hypothesis, “an unknown state is in a set  $\{|\phi_{\vec{k}}\rangle\}_{\vec{k} \in \mathbb{Z}_2^d}$ ” defined as

$$|\phi_{\vec{k}}\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{d_A}} \sum_i (-1)^{k_i} |i\rangle, \quad (40)$$

and an alternative hypothesis is a simple hypothesis “an unknown state is a pure state  $|\psi\rangle$ ”. An optimal success probability  $X_\epsilon(|\psi\rangle)$  of this problem is given as

$$\begin{aligned} X_\epsilon(|\psi\rangle) &\stackrel{\text{def}}{=} \max_T \{ \langle \psi | T | \psi \rangle \mid T \in \mathfrak{B}(\mathcal{H}), 0 \leq T \leq I, \\ & \quad \forall \vec{k} \in \mathbb{Z}_2^d, \langle \phi_{\vec{k}} | T | \phi_{\vec{k}} \rangle \leq \epsilon^2 \}, \end{aligned} \quad (41)$$

where  $d$  is the dimension of the Hilbert-space  $\mathcal{H}$ . Here, we define  $\epsilon$  so that  $\epsilon^2$  is an upper bound of the type 1 error. As we can easily see, this problem possesses a nice group symmetry; that is, our composite hypothesis is generated from a single state  $|\phi_0\rangle$  by a group action of *phase flipping*:  $|i\rangle \rightarrow -|i\rangle$ . Actually, we will use this property to derive an analytical formula of  $X_\epsilon(|\psi\rangle)$ .

In the next section, we will prove that this optimal success probability  $X_\epsilon(|\psi\rangle)$  is equal to the optimal success probability of the local hypothesis testing under separable POVM  $S_{\alpha, \text{sep}}(|\Psi\rangle)$  with just rescaling parameters:

$$X_{\sqrt{\alpha d_B}}(|\psi\rangle) = S_{\alpha, \text{sep}}(|\Psi\rangle), \quad (42)$$

where  $|\psi\rangle$  is defined as  $|\psi\rangle = \sum_{i=1}^{d_A} \sqrt{\lambda_i} |i\rangle$  by using the Schmidt coefficients  $\{\lambda_i\}_{i=1}^{d_A}$  of  $|\Psi\rangle$ . The aim of this subsection is deriving an analytical formula for  $X_\epsilon(|\psi\rangle)$  as a preparation to derive an analytical formula for  $S_{\alpha, \text{sep}}(|\Psi\rangle)$  by proving the above equality in the next section. Thus, we only treat a real  $|\psi\rangle$  in this subsection; that is,  $|\psi\rangle$  satisfies  $\langle i | \psi \rangle \in \mathbb{R}$  for all  $i$ . In this case, without losing generality, we can always assume  $\langle i | \psi \rangle \geq 0$  for all  $i$  by changing appropriate states in the basis as  $|i\rangle \rightarrow -|i\rangle$ . Moreover, by changing the label of the basis, without losing generality, we can also assume  $\langle i | \psi \rangle \geq \langle i+1 | \psi \rangle$  for all  $i$ . In the following discussion, we always choose the standard basis of  $\mathcal{H}_A$  as above.

### B. Reduction to a hypothesis testing with a simple alternative hypothesis

In this subsection, we show that the above global hypothesis testing with a composite null hypothesis can be reduced to a global hypothesis testing with an additional restriction on POVM.

First, we observe that an optimal  $T$  can be chosen as  $\text{rank} T_0 = 1$ .

*Lemma 5:*

$$\begin{aligned} & X_\epsilon(|\psi\rangle) \\ &= \max_T \{ \langle \psi | T | \psi \rangle \mid T \in \mathfrak{B}(\mathcal{H}), 0 \leq T \leq I, T = \text{Re} T, \\ & \quad \text{rank} T = 1, \forall \vec{k} \in \mathbb{Z}_2^d, \langle \phi_{\vec{k}} | T | \phi_{\vec{k}} \rangle \leq \epsilon^2 \} \end{aligned} \quad (43)$$

*Proof:* About the condition  $T = \text{Re} T$ , we have already seen that this condition does not change the value of the optimization problem at the last of the previous subsection. Thus, here, we only treat the condition  $\text{rank} T = 1$ .

When  $X_\epsilon(|\psi\rangle) = 0$ , we can always choose  $T = 0$ , which satisfies  $\text{rank} T = 1$ . Thus, we assume  $X_\epsilon(|\psi\rangle) > 0$ , and, hence,  $\langle \psi | T | \psi \rangle > 0$  for an optimal  $T$ .

Suppose  $T_0$  is an optimal POVM and there exists a state  $|\psi^\perp\rangle \in \text{Ran} T_0$  satisfying  $\langle \psi^\perp | \psi \rangle = 0$ . Then, from  $\det T_0 > 0$  and the continuity of  $\det(T_0 - p|\psi^\perp\rangle\langle\psi^\perp|)$  with respect to  $p$ , there



exists a minimum  $p > 0$  satisfying  $\det(T_0 - p|\psi^\perp\rangle\langle\psi^\perp|) > 0$ , where a determinant is defined only on  $\text{Ran}T_0$ . We call this optimal  $p$  as  $p_0$ . Then, a positive operator  $T'_0 \stackrel{\text{def}}{=} T_0 - p_0|\psi^\perp\rangle\langle\psi^\perp|$  satisfies  $\text{Ran}T'_0 = \text{Ran}T_0 - 1$ ,  $\langle\psi|T'_0|\psi\rangle = \langle\psi|T_0|\psi\rangle$  and  $\langle\phi_{\vec{k}}|T'_0|\phi_{\vec{k}}\rangle = \langle\phi_{\vec{k}}|T_0|\phi_{\vec{k}}\rangle - p_0|\langle\phi_{\vec{k}}|\psi^\perp\rangle|^2 \leq \epsilon^2$ . Thus,  $T'_0$  is an optimal POVM whose range does not include a state  $|\psi^\perp\rangle$ .

By repeating the above argument, we can conclude that there exists an optimal POVM  $T_0$  whose range does not include any state  $|\psi^\perp\rangle$  satisfying  $\langle\psi^\perp|\psi\rangle = 0$ . This optimal POVM  $T_0$  should satisfy  $\text{rank}T_0 = 1$ . We will show this fact by contradiction. Suppose  $\text{rank}T_0 \geq 2$  for this  $T_0$ . Then, there exist states  $|e_0\rangle, |e_1\rangle \in \text{Ran}T_0$  satisfying  $\langle e_0|\psi\rangle \neq 0$ ,  $\langle e_0|e_1\rangle = 0$ . Then, we can write down these states as

$$\begin{aligned} |e_0\rangle &= \alpha_0|\psi\rangle + \sqrt{1 - |\alpha_0|^2}|\psi_0^\perp\rangle \\ |e_1\rangle &= \alpha_1|\psi\rangle + \sqrt{1 - |\alpha_1|^2}|\psi_1^\perp\rangle, \end{aligned}$$

where  $\alpha_0 \neq 0$ , and the states  $|\psi_0^\perp\rangle$  and  $|\psi_1^\perp\rangle$  satisfies  $\langle\psi_0^\perp|\psi\rangle = \langle\psi_1^\perp|\psi\rangle = 0$ . Then, we can conclude that an operator  $|\psi^\perp\rangle$  defined as

$$\begin{aligned} |\psi^\perp\rangle &\stackrel{\text{def}}{=} \alpha_1|e_0\rangle - \alpha_0|e_1\rangle \\ &= \alpha_1\sqrt{1 - |\alpha_0|^2}|\psi_0^\perp\rangle - \alpha_0\sqrt{1 - |\alpha_1|^2}|\psi_1^\perp\rangle \end{aligned}$$

satisfies  $|\psi^\perp\rangle \neq 0$ . Since  $|\psi^\perp\rangle \in \text{Ran}T_0$  and  $\langle\psi^\perp|\psi\rangle = 0$ , this is a contradiction.  $\square$

From the previous lemma, we can always choose an optimal POVM  $T$  as  $T = |\phi\rangle\langle\phi|$ . Moreover, from non-negativity of  $\langle i|\psi\rangle$ , we can also assume  $\langle i|\phi\rangle \geq 0$  for all  $i$  as follow:

*Lemma 6:*

$$\begin{aligned} X_\epsilon(|\psi\rangle) &= \max_{|\phi\rangle} \{ \langle\psi|\phi\rangle^2 \mid |\phi\rangle \in \mathcal{H}, \|\psi\|^2 \leq 1, \\ &\quad \forall i, \langle i|\psi\rangle \geq 0, \forall \vec{k} \in \mathbb{Z}_2^d, |\langle\phi_{\vec{k}}|\phi\rangle|^2 \leq \epsilon^2 \}. \end{aligned}$$

*Proof:* First, we can always choose an optimal state  $|\phi\rangle$  as  $\langle\psi|\phi\rangle \geq 0$ . We define coefficients  $b_i$  as  $|\phi\rangle = \sum_i b_i|i\rangle$ . Suppose there exists  $i_0$  such that  $b_{i_0} < 0$  for an optimal  $|\phi\rangle$  satisfying  $\langle\psi|\phi\rangle \geq 0$ . We define  $|\phi'\rangle = \sum_i b'_i|i\rangle$  as  $b'_{i_0} = -b_{i_0}$  and  $b'_i = b_i$  for all  $i \neq i_0$ . Then,  $\langle\phi_{\vec{k}}|\phi'\rangle \leq \epsilon^2$  for all  $\vec{k}$  guarantees  $\langle\phi_{\vec{k}}|\phi'\rangle \leq \epsilon^2$ , and  $\langle i|\psi\rangle \geq 0$  for all  $i$  guarantees  $\langle\psi|\phi'\rangle > \langle\psi|\phi\rangle$ . This is a contradiction. Therefore, an optimal  $|\phi\rangle$  satisfying  $\langle\psi|\phi\rangle \geq 0$  must satisfy  $\langle i|\phi\rangle \geq 0$  for all  $i$ . In other words, we can always choose an optimal state  $|\psi\rangle$  as above.  $\square$

Finally, we can transform  $X_\epsilon(|\psi\rangle)$  in the following form:

*Lemma 7:*

$$\begin{aligned} X_\epsilon(|\psi\rangle) &= \left[ \max \{ \langle\psi|\phi\rangle \mid |\phi\rangle \in \mathcal{H}, \|\phi\|^2 \geq 1, \right. \\ &\quad \left. \forall i, \langle i|\phi\rangle \geq \langle i+1|\phi\rangle \geq 0, \langle\phi_d|\phi\rangle \leq \epsilon \} \right]^2, \end{aligned} \quad (44)$$

where  $|\phi_j\rangle$  is defined as

$$|\phi_j\rangle = \frac{1}{\sqrt{j}} \sum_{i=1}^j |i\rangle. \quad (45)$$

*Proof:* From the previous lemma, we can always choose an optimal state  $|\phi\rangle = \sum_i b_i|i\rangle$  as one satisfying  $b_i \geq 0$  for

all  $i$ . Suppose there exists a pair  $i_0 < i_1$  such that  $b_{i_0} < b_{i_1}$ . We define  $|\phi'\rangle = \sum_i b'_i|i\rangle$  as  $b'_{i_0} = b_{i_1}$ ,  $b'_{i_1} = b_{i_0}$ , and  $b'_i = b_i$  for all  $i \neq i_0, i_1$ . Then,  $|\phi'\rangle$  satisfies  $|\langle\phi_{\vec{k}}|\phi'\rangle|^2 \geq \epsilon^2$  for all  $\vec{k} \in \mathbb{Z}_2^d$  and  $\langle\psi|\phi'\rangle > \langle\psi|\phi\rangle$ . Thus,  $|\psi\rangle$  is not an optimal state; this is a contradiction. Therefore, an optimal state  $|\phi\rangle$  with  $b_i \geq 0$  satisfies  $b_i \geq b_{i+1}$  for all  $i$ . This optimal state apparently satisfies

$$\langle\phi_d|\phi\rangle^2 \geq |\langle\phi_{\vec{k}}|\phi\rangle|^2. \quad (46)$$

Thus, we can replace the condition  $|\langle\phi_{\vec{k}}|\phi\rangle|^2 \leq \epsilon^2$  by the condition  $\langle\phi_d|\phi\rangle \leq \epsilon$  for this optimal state.  $\square$

The optimization problem in Eq.(44) does not contain a composite hypothesis, but is a hypothesis testing of two simple hypotheses  $|\psi\rangle$  and  $|\phi_d\rangle$  with an additional restriction on the form of the POVM. In the next subsection, we analytically solve this optimization problem.

### C. Derivation of analytical solutions of $X_\epsilon(|\psi\rangle)$

First, we give solutions of  $X_\epsilon(|\psi\rangle)$  for two trivial cases. When  $|\psi\rangle = |\phi_d\rangle$ , we can easily see  $X_\epsilon(|\phi_d\rangle) = \epsilon$ . When  $\langle\psi|\phi_d\rangle \leq \epsilon$ , we can choose an optimal vector  $|\phi\rangle$  as  $|\phi\rangle = |\psi\rangle$ . Hence,  $X_\epsilon(|\phi_d\rangle) = 1$ .

For  $|\psi\rangle \neq |\phi_d\rangle$ , we derive the following lemma:

*Lemma 8:* Suppose  $d \geq 2$ ,  $|\psi\rangle \neq |\phi_d\rangle$ , and  $|\phi\rangle$  attains the optimal of Eq.(44). Then, at least, one of the following two statement is true:

- 1)  $|\phi\rangle \in \text{span}\{|\psi\rangle, |\phi_d\rangle\}$ ,  $\|\phi\| = 1$ .
- 2) There exists an optimal state  $|\phi'\rangle$  satisfying  $\langle d|\phi'\rangle = 0$ .

*Proof:* Suppose  $|\phi\rangle$  attains the optimal of Eq.(44). First, we can uniquely decompose  $|\phi\rangle$  as follows:

$$|\phi\rangle = \alpha|\psi\rangle + \beta|\phi_d\rangle + |y\rangle, \quad (47)$$

where  $\alpha$  and  $\beta$  are real numbers, and  $|y\rangle$  satisfies  $|y\rangle \perp |\phi_d\rangle$  and  $|y\rangle \perp |\psi\rangle$ . Then, we define a Schmidt orthogonalized state  $|\phi_d^\perp\rangle$  on a subspace  $\text{span}\{|\psi\rangle, |\phi_d\rangle\}$  as

$$|\phi_d^\perp\rangle \stackrel{\text{def}}{=} \frac{|\psi\rangle - c|\phi_d\rangle}{\| |\psi\rangle - c|\phi_d\rangle \|}, \quad (48)$$

where  $c \stackrel{\text{def}}{=} \langle\psi|\phi_d\rangle$ . By the definition,  $|\phi_d^\perp\rangle$  satisfies  $\langle\phi_d^\perp|\psi\rangle > 0$ . Also, by defining the coefficients  $\{\xi_i\}_{i=1}^d$  as  $|\phi_d^\perp\rangle = \sum_i \xi_i|i\rangle$ , these coefficients satisfy  $\xi_i \geq \xi_{i+1}$  for all  $i$ . Moreover, the fact  $\langle\phi_d^\perp|\phi_d\rangle = 0$  guarantees that there exists a natural number  $l$  satisfying  $\xi_l \geq 0 > \xi_{l+1}$ .

First, we consider the case when  $\|\phi\| < 1$ . In this case, actually,  $|\phi\rangle$  satisfies  $\langle d|\phi\rangle = 0$ . This fact can be proven by contradiction as follows: Suppose  $\langle d|\phi\rangle > 0$ . Then, we can choose a small number  $\delta > 0$  such that a vector  $|\phi'\rangle \stackrel{\text{def}}{=} |\phi\rangle + \delta|\phi_d^\perp\rangle$  satisfies  $\|\phi'\| \leq 1$ ,  $\langle\phi_d|\phi'\rangle = \langle\phi_d|\phi\rangle$ ,  $\langle i|\phi'\rangle \geq 0$  for all  $i$ , and  $\langle\psi|\phi'\rangle = \langle\psi|\phi\rangle + \alpha\langle\psi|\phi_d^\perp\rangle > \langle\psi|\phi\rangle$ . Thus,  $|\phi\rangle$  is not an optimal state. This is a contradiction. Hence, in this case, a state  $|\phi\rangle$  itself is an optimal state satisfying the condition  $\langle d|\phi\rangle = 0$ .

Second, we consider the case  $\|\phi\| = 1$ . In this case, we consider the case  $\alpha < 0$  and the case  $\alpha \geq 0$  separately. Thus, we consider the case when  $\|\phi\| = 1$  and  $\alpha < 0$ . In this case,  $\langle d|\phi\rangle = 0$  is proven by contradiction as follows: Suppose

$\langle d|\phi\rangle = 0$ . Then, we can choose a real number  $\gamma$  so that it satisfies

$$0 \leq \gamma \leq -\alpha\|\psi\rangle - c|\phi_d\rangle\|, \quad (49)$$

and a vector  $|\phi'\rangle \stackrel{\text{def}}{=} \gamma|\phi_d^\perp\rangle + |\phi\rangle$  satisfies  $\langle d|\phi'\rangle \geq 0$ . This vector  $|\phi'\rangle$  satisfies  $\langle\phi_d|\phi'\rangle = \langle\phi_d|\phi\rangle$ ,  $\langle i|\phi'\rangle \geq \langle i+1|\phi'\rangle \geq 0$  for all  $i$ , and  $\langle\psi|\phi'\rangle = \gamma\langle\psi|\phi_d^\perp\rangle + \langle\psi|\phi\rangle > \langle\psi|\phi\rangle$ . Furthermore, from Eq.(49) and  $\|\phi\rangle = 1$ , the formula

$$|\phi'\rangle = (c + \beta)|\phi_d\rangle + (\alpha\|\psi\rangle - c|\phi_d\rangle + \gamma)|\phi_d^\perp\rangle + |y\rangle \quad (50)$$

guarantees  $\|\phi'\rangle\| < 1$ . Thus,  $|\phi\rangle$  is not an optimal. This is a contradiction. Therefore, we have  $\langle d|\phi\rangle = 0$  in this case.

Next, we consider the case when  $\|\phi\rangle = 1$  and  $\alpha \geq 0$ . We define a vector  $|x\rangle$  as  $|x\rangle \stackrel{\text{def}}{=} \alpha|\psi\rangle + \beta|\phi_d\rangle$ , and its coefficients  $\{x_i\}_{i=1}^d$  as  $|x\rangle = \sum_{i=1}^d x_i|i\rangle$ , which apparently satisfy  $x_i \geq x_{i+1}$  for all  $i$ . In this case, we also consider the cases  $x_d = 0$ ,  $x_d > 0$ , and  $x_d < 0$  separately.

First, we consider the case  $x_d = 0$ . In this case, this vector  $|x\rangle$  is apparently an optimal vector satisfying  $\langle d|x\rangle = 0$ .

Second, we consider the case  $x_d > 0$ . In this case, this vector  $|x\rangle$  is apparently an optimal vector satisfying  $|x\rangle \in \text{span}\{|\psi\rangle, |\phi_d\rangle\}$ . Moreover, we can prove that this vector also satisfies  $\|x\rangle = 1$  by contradiction as follows: Suppose  $\|x\rangle < 1$ . Then, there exists a small number  $\delta > 0$  such that a new vector  $|x'\rangle \stackrel{\text{def}}{=} \alpha|\psi\rangle + \beta|\phi_d\rangle + \delta|\phi_d^\perp\rangle$  satisfying  $\|x'\rangle \leq 1$  and  $\langle d|x'\rangle \geq 0$ . This vector  $|x'\rangle$  satisfies  $\langle\phi_d|x'\rangle = \langle\phi_d|x\rangle$ ,  $\langle i|x'\rangle \geq \langle i+1|x'\rangle \geq 0$  for all  $i$ , and  $\langle\psi|x'\rangle = \langle\psi|x\rangle + \delta\langle\psi|\phi_d^\perp\rangle > \langle\psi|x\rangle$ . Thus,  $|\phi\rangle$  is not an optimal vector. This is contradiction. Therefore,  $\|x\rangle = 1$ , and this means  $|x\rangle = |\phi\rangle$ . Hence,  $|\phi\rangle \in \text{span}\{|\psi\rangle, |\phi_d\rangle\}$ .

Finally, we consider the case when  $x_d < 0$ . In this case, there exists a natural number  $m \leq d+1$  such that  $x_m \geq 0 > x_{m+1}$ . Here, we define a one-parameter family of vectors  $\{|z_\delta\rangle\}_{0 \leq \delta \leq 1}$  as  $|z_\delta\rangle \stackrel{\text{def}}{=} |x\rangle + \delta|y\rangle$ . Hence,  $|z_0\rangle = |x\rangle$  and  $|z_1\rangle = |\phi\rangle$ . For all  $0 \leq \delta \leq 1$ , this family satisfies  $\langle\psi|z_\delta\rangle = \langle\psi|x\rangle = \langle\psi|\phi\rangle$ ,  $\langle\phi_d|z_\delta\rangle = \langle\phi_d|x\rangle = \langle\phi_d|\phi\rangle$ , and  $\|z_\delta\rangle \leq \|\phi\rangle = 1$ . We define a function  $f(\delta)$  as  $f(\delta) = (z_{\delta,1}, \dots, z_{\delta,d})$ , where  $z_{\delta,i}$  is defined as  $|z_\delta\rangle = \sum_{i=1}^d z_{\delta,i}|i\rangle$ . Then, the point  $(z_{1,1}, \dots, z_{1,d})$  satisfies  $z_{1,i} \geq 0$  for all  $i$ , and the point  $(z_{0,1}, \dots, z_{0,d})$  satisfies  $z_{0,i} \geq 0$  for  $i \leq m$  and  $z_{0,i} < 0$  for  $i \geq m+1$ . Hence, a connecting curve  $f(\delta)$  on the  $d$ -dimensional space starts from the outside and goes into the region  $\{x_i \geq 0, \forall i\}$ . Therefore, this curve must across the boundary of this region in somewhere between the start point  $\delta = 0$  and the end point  $\delta = 1$ . Thus, there exists  $0 < \delta_0 \leq 1$  such that  $z_{\delta_0,i} \geq 0$  for all  $i$  and there exists  $i_0$  satisfying  $z_{\delta_0,i_0} = 0$ . Next, we define new coefficients  $\{z'_i\}_{i=1}^d$  which are derived by reordering  $\{z_{\delta_0,i}\}_{i=1}^d$  so that they satisfy  $z'_i \geq z'_{i+1} \geq 0$ . Then, a state  $|\phi'\rangle$  defined as  $|\phi'\rangle \stackrel{\text{def}}{=} \sum_i z'_i|i\rangle$  satisfies  $\langle\psi|\phi'\rangle \geq \langle\psi|\phi\rangle$ ,  $\langle\phi_d|\phi'\rangle = \langle\phi_d|\phi\rangle$ ,  $\|\phi'\rangle \leq \|\phi\rangle$ ,  $\langle i|\phi'\rangle \geq 0$ . Therefore,  $|\phi'\rangle$  is actually an optimal state satisfying  $\langle d|\phi'\rangle = 0$ .  $\square$

The following lemma gives a non-trivial solution of the optimization problem.

**Lemma 9:** Consider the case when  $d \geq 2$  and  $\epsilon < \langle\phi_d|\psi\rangle <$

1. Suppose  $|\phi\rangle$  defined as

$$|\phi\rangle \stackrel{\text{def}}{=} \frac{\sqrt{1-\epsilon^2}|\psi\rangle - (c\sqrt{1-\epsilon^2} - \epsilon\sqrt{1-c^2})|\phi_d\rangle}{\sqrt{1-c^2}}, \quad (51)$$

where  $c \stackrel{\text{def}}{=} \langle\psi|\phi_d\rangle$ , satisfies  $\langle d|\phi\rangle \geq 0$ . Then,  $|\phi\rangle$  attains the optimum of Eq.(44).

*Proof:* We define a new function  $X'_\epsilon(|\psi\rangle)$  as follows:

$$\begin{aligned} X'_\epsilon(|\psi\rangle) \\ = \max_{|\phi\rangle} \{ \langle\psi|\phi\rangle^2 \mid |\phi\rangle \in \mathcal{H}, \|\psi\rangle\|^2 \leq 1, |\langle\phi_d|\phi\rangle|^2 \leq \epsilon^2 \}. \end{aligned}$$

Then, by the definition,  $X'_\epsilon(|\psi\rangle)$  satisfies  $X_\epsilon(|\psi\rangle) \leq X'_\epsilon(|\psi\rangle)$ . Thus, if  $|\phi\rangle$  defined by Eq.(51) attains the optimum of  $X'_\epsilon(|\psi\rangle)$ , and also satisfies  $\langle d|\phi\rangle \geq 0$ , this vector  $|\phi\rangle$  apparently also attains the optimum of  $X_\epsilon(|\psi\rangle)$ . In the remaining part of this proof, we prove actually this is the case; this  $|\phi\rangle$  is an optimal vector for  $X'_\epsilon(|\psi\rangle)$ .

Suppose  $|\phi\rangle$  is an optimal vector of Eq.(53). Then, by the definition of  $X'_\epsilon(|\psi\rangle)$ ,  $|\phi\rangle$  is apparently on the subspace  $\text{span}\{|\psi\rangle, |\phi_d\rangle\}$ . Thus, we define  $r$ ,  $\theta$  and  $\xi$  satisfying  $r > 0$ ,  $-\pi \leq \theta \leq \pi$  and  $-\pi \leq \xi \leq \pi$ , respectively as

$$\begin{aligned} |\psi\rangle &= \cos\theta|\phi_d\rangle + \sin\theta|\phi_d^\perp\rangle \\ |\phi\rangle &= r(\cos\xi|\phi_d\rangle + \sin\xi|\phi_d^\perp\rangle), \end{aligned} \quad (52)$$

where  $|\phi_d^\perp\rangle$  is defined by Eq. (48). By the definitions, we have  $\cos\theta = \langle\psi|\phi_d\rangle$ ,  $\sin\theta = \langle\psi|\phi_d^\perp\rangle$ ,  $\cos\xi = \langle\psi|\phi_d\rangle/r$ , and  $\sin\xi = \langle\psi|\phi_d^\perp\rangle/r$ . Thus,  $\langle\psi|\phi_d\rangle > 0$  and  $\langle\psi|\phi_d^\perp\rangle > 0$  guarantee  $0 < \theta < \frac{\pi}{2}$ .  $\langle\psi|\phi\rangle > 0$  guarantees  $-\frac{\pi}{2} \leq \theta - \xi \leq \frac{\pi}{2}$ .

First, we prove  $\xi \geq 0$  by contradiction. Suppose  $\xi < 0$ ; that is,  $-\frac{\pi}{2} \leq \xi < 0$ . Then, defining  $|\phi'\rangle$  by using  $\xi' \stackrel{\text{def}}{=} -\xi$  instead of  $\xi$  in Eq.(52), we have  $|\langle\phi_d|\phi'\rangle| \leq \epsilon$ ,  $\|\phi'\rangle = \|\phi\rangle \leq 1$ . Moreover, the inequalities  $|\xi' - \theta| < |\xi'| + |\theta| = -\xi + \theta \leq \frac{\pi}{2}$  guarantee  $\langle\psi|\phi'\rangle > \langle\psi|\phi\rangle$ . Thus,  $|\phi\rangle$  is not optimal; this is contradiction. Therefore,  $\xi$  satisfies  $\xi \geq 0$ .

Second, we prove  $r = 1$  by contradiction. Suppose  $r < 1$ . Then, we can choose a small number  $\delta > 0$  such that a state  $|\phi'\rangle$  defined as  $|\phi'\rangle \stackrel{\text{def}}{=} |\phi\rangle + \delta|\phi_d^\perp\rangle$  satisfies  $\|\phi'\rangle \leq 1$ . Then, this state  $|\phi'\rangle$  satisfies  $|\langle\phi_d|\phi'\rangle| = |\langle\phi_d|\phi\rangle| \leq \epsilon$ , and  $\langle\psi|\phi'\rangle > \langle\psi|\phi\rangle$ . Thus,  $|\phi\rangle$  is not optimal; this is contradiction. Therefore,  $r$  satisfies  $r = 1$ .

Finally, we prove  $\epsilon = \langle\phi_d|\phi\rangle$  by contradiction. Suppose  $\epsilon > \langle\phi_d|\phi\rangle = \cos\xi$ . In this case, we can choose a small number  $\delta > 0$  such that  $\xi' \stackrel{\text{def}}{=} \xi - \delta$  satisfies  $|\cos\xi'| \leq \epsilon$ . Then, defining  $|\phi'\rangle$  by using  $\xi'$  and  $r = 1$  in Eq.(52), we derive  $\langle\psi|\phi'\rangle = \cos(\theta - \xi') > \cos(\theta - \xi) = \langle\psi|\phi\rangle$ . Thus,  $|\phi\rangle$  is not optimal; this is contradiction. Therefore, an optimal vector  $|\phi\rangle$  is the unique vector satisfying  $\epsilon = \langle\phi_d|\phi\rangle$  and  $0 \leq \xi \leq \frac{\pi}{2} + \theta \leq \pi$ . Eq.(52) guarantees that this vector  $|\phi\rangle$  can be written in the form of Eq.(51).  $\square$

At the next step, by means of Lemma 8 and Lemma 9 we derive the following lemma:

**Lemma 10:** Suppose  $d \geq 2$  and  $\epsilon < \langle\phi_d|\psi\rangle < 1$ . Define  $|\phi\rangle$  as Eq.(51). Then, when  $\langle d|\phi\rangle \geq 0$ ,  $|\phi\rangle$  is an optimal vector for Eq.(44), and when  $\langle d|\phi\rangle < 0$ , there exists an optimal vector  $|\phi'\rangle$  satisfying  $\langle d|\phi'\rangle = 0$  for Eq.(44).

*Proof :* We first consider the case where the optimal vector  $|\phi\rangle$  satisfies  $|\phi\rangle \in \text{span}\{|\psi\rangle, |\phi_d\rangle\}$  and  $\| |\phi\rangle \| = 1$ . In this case, we can define notations as Eq(52) in the last section again. Here, we choose  $\theta$  and  $\xi$  to be  $-\pi < \theta \leq \pi$  and  $-\pi < \xi \leq \pi$  for convenience. By the definitions, we again have  $0 < \theta < \frac{\pi}{2}$  and  $-\frac{\pi}{2} \leq \theta - \xi \leq \frac{\pi}{2}$ . In the similar way,  $\langle \phi | \phi_d \rangle > 0$  guarantees  $-\frac{\pi}{2} \leq \xi \leq \frac{\pi}{2}$  in this case.

First, we prove  $\xi > 0$  by contradiction. Suppose  $\xi \leq 0$ . As we explained in the proof of Lemma 8, there exist a natural number  $l \leq d-1$  such that  $\langle l+1 | \phi_d^\perp \rangle < 0$ . From this fact and Eq.(52),  $\xi \leq 0$  guarantees  $\langle l+1 | \phi \rangle < 0$ . Thus,  $|\phi\rangle$  is not an optimal vector of Eq.(44). This is contradiction. Therefore,  $\xi$  satisfies  $0 < \xi \leq \frac{\pi}{2}$ .

Second, we can prove  $\theta < \xi$  in the completely same discussion of the previous lemma. Therefore,  $\theta$  and  $\xi$  satisfy  $0 < \theta < \xi \leq \frac{\pi}{2}$ . When  $|\psi\rangle$  satisfies  $\langle \phi | \phi_d \rangle = \epsilon$ ,  $|\psi\rangle$  can be written down as Eq.(51). Thus, since we are assuming the optimality of  $|\psi\rangle$ ,  $|\psi\rangle$  satisfies  $\langle d | \psi \rangle \geq 0$ . On the other hand, when  $|\psi\rangle$  satisfies  $\langle \phi | \phi_d \rangle < \epsilon$ , we can prove  $\langle d | \phi \rangle = 0$  by contradiction. Suppose  $\langle d | \phi \rangle > 0$  and  $\langle \phi | \phi_d \rangle < \epsilon$ . Then, we can choose a small number  $\delta$  such that a vector  $|\phi'\rangle \stackrel{\text{def}}{=} \cos(\xi - \delta)|\phi_d\rangle + \sin(\xi - \delta)|\phi_d^\perp\rangle$  satisfies  $\langle \phi_d | \phi' \rangle \leq \epsilon$ , and  $\langle i | \phi' \rangle \geq 0$  for all  $i$ . Since  $|\phi'\rangle$  satisfies  $\langle \psi | \phi' \rangle > \langle \psi | \phi \rangle$ ,  $|\phi\rangle$  is not optimal; this is contradiction. Therefore, when  $\langle \phi_d | \phi \rangle < \epsilon$ ,  $|\phi\rangle$  satisfies  $\langle d | \phi \rangle = 0$ .

Now, we consider the case there is no agreement whether an optimal vector  $|\phi\rangle$  satisfies  $|\phi\rangle \in \text{span}\{|\psi\rangle, |\phi_d\rangle\}$ , or not. When  $|\phi\rangle$  defined by Eq.(51) satisfies  $\langle d | \phi \rangle \geq 0$ , then from Lemma 9, this vector  $|\phi\rangle$  is an optimal vector. Then, we consider the case when  $|\phi\rangle$  defined by Eq.(51) does not satisfy  $\langle d | \phi \rangle \geq 0$ . In this case, if there exists an optimal vector  $|\phi\rangle$  on the subspace  $\text{span}\{|\psi\rangle, |\phi_d\rangle\}$  satisfying  $\| |\psi\rangle \| = 1$ ,  $|\phi\rangle$  should satisfy  $\langle \phi | \phi_d \rangle < \epsilon$ , and thus,  $\langle d | \phi \rangle = 0$  from the above discussion. Otherwise, there is no optimal vector satisfying  $\| |\phi\rangle \| = 1$  on the subspace  $\text{span}\{|\psi\rangle, |\phi_d\rangle\}$ . In this case, from Lemma 8, there exists an optimal vector  $|\phi'\rangle$  satisfying  $\langle d | \phi' \rangle = 0$ . Therefore, the statement of the present lemma is true.  $\square$

Finally, from the above lemma, we derive the following theorem, which gives a complete analytical formula for the optimal success probability  $X_\epsilon(|\psi\rangle)$  and the optimal strategy of the global hypothesis testing considering in this section:

**Theorem 4:** Suppose  $|\psi\rangle \in \mathcal{H}$  can be written down as  $|\psi\rangle = \sum_i \sqrt{\lambda_i} |i\rangle$ . Define a natural number  $\eta$  as

$$\eta \stackrel{\text{def}}{=} \min_{l \in \mathbb{N}} \left\{ l \mid l \leq d, \epsilon_l < \langle \phi_l | \psi_l \rangle, |\psi_l\rangle \neq |\phi_l\rangle, \langle l | \phi'_l \rangle < 0 \right\} - 1. \quad (53)$$

In the above formula,  $\epsilon_l$  is defined as  $\epsilon_l \stackrel{\text{def}}{=} \sqrt{\frac{d}{l}} \epsilon$ , a state  $|\psi_l\rangle$  is defined as

$$|\psi_l\rangle \stackrel{\text{def}}{=} \sum_{i=1}^l \sqrt{\lambda_i} |i\rangle / \sqrt{\sum_{i=1}^l \lambda_i}, \quad (54)$$

a state  $|\phi_l\rangle$  is defined by Eq.(45), and a state  $|\phi'_l\rangle$  is defined

as

$$|\phi'_l\rangle = \frac{\sqrt{1 - \epsilon_l^2} |\psi_l\rangle - (c_l \sqrt{1 - \epsilon_l^2} - \epsilon_l \sqrt{1 - c_l^2}) |\phi_l\rangle}{\sqrt{1 - c_l^2}}, \quad (55)$$

where  $c_l$  is defined as  $c_l \stackrel{\text{def}}{=} \langle \psi_l | \phi_l \rangle$ . Then,  $\eta$  satisfies  $\eta \geq 1$ , and the following statements are true:

- 1) In the case when  $\epsilon_\eta \geq \langle \phi_\eta | \psi_\eta \rangle$ ,

$$X_\epsilon(|\psi\rangle) = \sum_{i=1}^{\eta} \lambda_i, \quad (56)$$

and a state  $|\phi\rangle = |\psi_\eta\rangle$  attains the optimum.

- 2) In the case when  $\epsilon_\eta < \langle \phi_\eta | \psi_\eta \rangle$ ,  $X_\epsilon(|\psi\rangle)$  is given as

$$X_\epsilon(|\psi\rangle) = \left( \sum_{i=1}^{\eta} \lambda_i \right) \cdot \left( \sqrt{1 - \epsilon_\eta^2} \sqrt{1 - c_\eta^2} + \epsilon_\eta c_\eta \right)^2. \quad (57)$$

A vector  $|\phi\rangle$  attaining this optimum is given as  $|\phi\rangle = \epsilon_\eta |\phi_\eta\rangle$  in the case  $|\psi_\eta\rangle = |\phi_\eta\rangle$ , and  $|\phi\rangle = |\phi'_\eta\rangle$  in the case  $|\psi_\eta\rangle \neq |\phi_\eta\rangle$ , respectively

Here, we add one remark: When  $\eta \geq 2$ ,  $\epsilon_\eta \leq \langle \phi_\eta | \psi_\eta \rangle$  and  $|\psi_\eta\rangle = |\phi_\eta\rangle$ , by redefining  $|\phi\rangle = \epsilon_\eta |\phi_\eta\rangle + \sqrt{1 - \epsilon_\eta^2} |\phi_\eta^\perp\rangle$ , we can make  $|\phi\rangle$  be a *normalized vector*. Therefore, in the case  $\eta \geq 2$ , we can always choose  $|\phi\rangle$  as a normalized vector; that is,  $T$  is a pure state.

*Proof :* Suppose the formula

$$X_\epsilon(|\psi\rangle) = \left( \sum_{i=1}^l \lambda_i \right) \cdot X_{\epsilon_l}(|\psi_l\rangle) \quad (58)$$

holds for  $l = \eta$ . Then, in the case  $\langle \psi_\eta | \phi_\eta \rangle \leq \epsilon_\eta$ , since  $|\phi\rangle = \psi_\eta$  attains  $X_{\epsilon_\eta}(|\psi_\eta\rangle) = 1$ , we derive Eq.(56). In the case  $\langle \psi_\eta | \phi_\eta \rangle > \epsilon_\eta$ , from the definition of  $\eta$ , either  $|\psi_\eta\rangle = |\phi_\eta\rangle$  or  $\langle \eta | \phi'_\eta \rangle \geq 0$  holds. When  $|\psi_\eta\rangle = |\phi_\eta\rangle$ , a state  $|\phi\rangle = \epsilon_\eta |\phi_\eta\rangle$  attains the optimum  $X_{\epsilon_\eta}(|\phi_\eta\rangle) = \epsilon_\eta$ . Thus, Eq.(57) holds. When  $\langle \eta | \phi'_\eta \rangle \geq 0$  holds, from Lemma 9, a state  $|\phi'_\eta\rangle$  given by Eq.(55) attains the optimum and  $X_\epsilon(|\psi\rangle)$  is given by Eq.(57). Hence, all the statements hold under this assumption. Thus, in the remaining part of this proof, we concentrate on proving Eq.(58) for  $l = \eta$ .

Here, we prove Eq.(58) for all  $\eta \leq l \leq d$  by induction starting from  $l = d$ . For  $l = d$ , Eq.(58) trivially holds. Suppose Eq.(58) holds for  $l$  satisfying  $1 \leq \eta < l \leq d$ . Then, from the definition of  $\eta$ , we have  $\epsilon_l < \langle \phi_l | \psi_l \rangle$ ,  $|\psi_l\rangle \neq |\phi_l\rangle$ , and  $\langle l | \phi'_l \rangle < 0$ . Thus, from Lemma 10, there exists a state  $|\phi\rangle \in \text{span}\{|i\rangle\}_{i=1}^l$  satisfying  $\langle l | \phi \rangle = 0$  and attaining the optimum of  $X_{\epsilon_l}(|\psi_l\rangle)$ , which is define by the optimization problem only on  $\text{span}\{|i\rangle\}_{i=1}^l$ . Thus, in this case  $X_{\epsilon_l}(|\psi_l\rangle)$  can be rewritten

as

$$\begin{aligned}
& X_{\epsilon_l}(|\psi_l\rangle) \\
&= \left[ \max \left\{ \langle \psi_l | \phi \rangle \mid |\phi\rangle \in \text{span}\{|i\rangle\}_{i=1}^{l-1}, \|\phi\|^2 \geq 1, \right. \right. \\
&\quad \left. \left. \forall i, \langle i | \phi \rangle \geq \langle i+1 | \phi \rangle \geq 0, \langle \phi | \phi \rangle \leq \epsilon_l \right\} \right]^2 \\
&= \frac{\sum_{i=1}^{l-1} \lambda_i}{\sum_{i=1}^l \lambda_i} \cdot \left[ \max \left\{ \langle \psi_{l-1} | \phi \rangle \mid |\phi\rangle \in \text{span}\{|i\rangle\}_{i=1}^{l-1}, \right. \right. \\
&\quad \left. \left. \|\phi\|^2 \geq 1, \forall i, \langle i | \phi \rangle \geq \langle i+1 | \phi \rangle \geq 0, \right. \right. \\
&\quad \left. \left. \langle \phi_{l-1} | \phi \rangle \leq \epsilon_{l-1} \right\} \right]^2 \\
&= \frac{\sum_{i=1}^{l-1} \lambda_i}{\sum_{i=1}^l \lambda_i} \cdot X_{\epsilon_{l-1}}(|\psi_{l-1}\rangle), \tag{59}
\end{aligned}$$

where we used relations  $\sqrt{l}\langle \phi_l | \phi \rangle = \sqrt{l-1}\langle \phi_{l-1} | \phi \rangle$  and  $\sqrt{\sum_{i=1}^l \lambda_i} \langle \psi_l | \phi \rangle = \sqrt{\sum_{i=1}^{l-1} \lambda_i} \langle \psi_{l-1} | \phi \rangle$  in the second equality. Thus, from Eq.(59) and Eq.(58) for  $l$ , we derive Eq.(58) for  $l-1$ . Therefore, Eq.(58) holds for all  $\eta \leq l \leq d$ .  $\square$

## VI. HYPOTHESIS TESTING UNDER SEPARABLE OPERATIONS

In this section, we treat the local hypothesis testing under separable POVM, and gives a proof of Theorem 3. As we have predicted in the last section, the proof is completed by showing that the local hypothesis testing under separable POVM is essentially equivalent to the global hypothesis testing treated in the last section, which is simpler than the former.

The equivalence of these two hypothesis testing problems can be written as the following theorem in terms of their optimal success probabilities  $X_\epsilon(|\psi\rangle)$  and  $S_{\alpha, \text{Sep}}(|\Psi\rangle)$ :

**Theorem 5:** For a state  $|\Psi\rangle = \sum_{i=1}^{d_A} \sqrt{\lambda_i} |ii\rangle \in \mathcal{H}_{AB}$  and a state  $|\psi\rangle = \sum_{i=1}^{d_A} \sqrt{\lambda_i} |i\rangle \in \mathcal{H}_A$ ,

$$S_{\alpha, \text{Sep}}(|\Psi\rangle) = X_{\sqrt{\alpha d_B}}(|\psi\rangle). \tag{60}$$

Since we have already derived an analytical formula for  $X_{\sqrt{\alpha d_B}}(|\psi\rangle)$  in Theorem 4 of the last section, we can derive Theorem 3 by just substituting  $S_{\alpha, \text{Sep}}(|\Psi\rangle) = 1 - \beta_{|\Psi\rangle, \text{Sep}}(\alpha)$  instead of  $X_{\sqrt{\alpha d_B}}(|\psi\rangle)$  in Theorem 4. Therefore, a proof of Theorem 3 completely reduces to a proof of Theorem 5. Thus, we will concentrate on a proof of Theorem 5 in all the remaining part of this section. The proof of this theorem can be divided into two parts: In the first part, we show that  $X_{\sqrt{\alpha d_B}}(|\psi\rangle)$  is an upper bound of  $S_{\alpha, \text{Sep}}(|\Psi\rangle)$ , and, then, in the second part, we show that this upper bound is actually achievable by a separable POVM. Organization of this section is as follows: In the subsection A, we give an upper bound on  $S_{\alpha, \text{Sep}}(|\Psi\rangle)$ , and show that the separability condition of POVM in its definition can be replaced by a condition in terms of a function  $\chi(\rho)$ . Then, we investigate properties of  $\chi(\rho)$  in subsection B. Finally, in the subsection C, we complete the proof of Theorem 5 by using lemmas derived in the subsection A and B.

### A. Reduction of the problem by means of a twirling

In this subsection, we derive an upper bound of  $S_{\alpha, \text{Sep}}(|\Psi\rangle)$  by using a twirling, which is a well-known technique to reduce a number of parameters of an optimization problem in quantum information [70], [17], [18], [71], [72]. Here, we use the twirling operation introduced in the paper [47]. Without losing generality, we can choose a computational basis as the Schmidt basis of  $|\Psi\rangle$  so that  $|\Psi\rangle$  can be written as

$$|\Psi\rangle = \sum_{i=1}^{d_A} \sqrt{\lambda_i} |ii\rangle, \tag{61}$$

where  $\{\lambda_i\}_{i=1}^{d_A}$  is the Schmidt coefficients of  $|\Psi\rangle$ . We define a family of local unitary operators  $U_{\vec{\theta}}$  parametrized by  $\vec{\theta} = \{\theta_i\}_{i=1}^d$  as follows,

$$U_{\vec{\theta}} = \left( \sum_{j=1}^{d_A} e^{i\theta_j} |j\rangle\langle j| \right) \otimes \left( \sum_{k=1}^{d_A} e^{-i\theta_k} |k\rangle\langle k| \right). \tag{62}$$

Note that  $(\mathcal{H}_{AB}, U_{\vec{\theta}})$  is a unitary representation of the compact topological group  $\overbrace{U(1) \times \cdots \times U(1)}^{d_A}$ ; by means of a unitary representation of a compact topological group, we implement the "twirling" operation (the averaging over the compact topological group) for a state (or POVM) [73]. We write this twirling operation as  $\Gamma$ . Since by an action of twirling operation, a given state is projected to the subspace of all invariant elements of the group action [73], we can calculate  $\Gamma(T)$  for any operator  $T \in \mathfrak{B}(\mathcal{H}_{AB})$  as follows:

$$\begin{aligned}
& \Gamma(T) \\
& \stackrel{\text{def}}{=} \int_0^{2\pi} \cdots \int_0^{2\pi} U_{\vec{\theta}} T U_{\vec{\theta}}^\dagger d\theta_1 \cdots d\theta_d \\
&= \left( \sum_{j=1}^d |e_j\rangle\langle e_j| \otimes |f_j\rangle\langle f_j| \right) T \left( \sum_{j=1}^d |e_j\rangle\langle e_j| \otimes |f_j\rangle\langle f_j| \right) \\
&\quad + \sum_{j \neq k} (|e_j\rangle\langle e_j| \otimes |f_k\rangle\langle f_k|) T (|e_j\rangle\langle e_j| \otimes |f_k\rangle\langle f_k|).
\end{aligned}$$

Suppose  $Q$  is a *maximally correlated subspace* with respect to the computational basis:

$$Q \stackrel{\text{def}}{=} \text{span}\{|ii\rangle\}_{i=1}^{d_A}. \tag{63}$$

Then, the above equation guarantees that all states on  $Q$  including  $|\Psi\rangle$  are invariant under the action of  $\Gamma$ :

$$\rho \in \mathfrak{B}(Q) \implies \Gamma(\rho) = \rho. \tag{64}$$

Here, we note that every state  $\rho$  on  $Q$  is a so called *maximally correlated state* [71], [74].

Defining  $\overline{S}_{\alpha, \text{Sep}}(|\Psi\rangle)$  as

$$\begin{aligned}
& \overline{S}_{\alpha, \text{Sep}}(|\Psi\rangle) \\
& \stackrel{\text{def}}{=} \max \left\{ \langle \Psi | T | \Psi \rangle | \text{Tr} T \leq \alpha d_A d_B, 0 \leq T \leq I, T \in \text{SEP} \right\}, \tag{65}
\end{aligned}$$

where  $\text{SEP}$  is the set of all (positive) separable operators on  $\mathcal{H}$ , we can easily see

$$S_{\alpha, \text{Sep}}(|\Psi\rangle) \leq \overline{S}_{\alpha, \text{Sep}}(|\Psi\rangle). \tag{66}$$

We define a function  $\chi(\rho)$  for a positive operator  $\rho \in \mathcal{P}_+(Q)$  as

$$\chi(\rho) \stackrel{\text{def}}{=} \min\{\text{Tr}(\rho + \sigma) \mid \exists \sigma = \sum_{j \neq k} p_{jk} |j\rangle\langle j| \otimes |k\rangle\langle k|, \\ 0 \leq \sigma \leq I, \rho + \sigma \in \text{SEP}\}. \quad (67)$$

Then, we can show the following lemma:

*Lemma 11:*

$$\bar{S}_{\alpha, \text{Sep}}(|\Psi\rangle) = \max \left\{ \langle \Psi | T_0 | \Psi \rangle \mid T_0 \in \mathfrak{B}(Q), \right. \\ \left. \chi(T_0) \leq \alpha d_A d_B, 0 \leq T_0 \leq I_Q \right\}, \quad (68)$$

where  $I_Q$  is an identity operator of the space  $Q$ :  $I_Q \stackrel{\text{def}}{=} \sum_{i=1}^{d_A} |ii\rangle\langle ii|$ .

*Proof:* Suppose  $T \in \mathfrak{B}(\mathcal{H}_{AB})$  is optimal for  $\bar{S}_{\alpha, \text{Sep}}(|\Psi\rangle)$ . Then, from (64), we can easily show  $\Gamma(T)$  is also optimal. On the other hand,  $\Gamma(T)$  can be written as

$$\Gamma(T) = T_0 + \sigma, \quad (69)$$

where  $T_0 \in \mathfrak{B}(Q)$  and  $\sigma$  can be written as  $\sigma = \sum_{j \neq k} p_{jk} |j\rangle\langle j| \otimes |k\rangle\langle k|$ . Thus, we have

$$\begin{aligned} & \bar{S}_{\alpha, \text{Sep}}(|\Psi\rangle) \\ &= \max \left\{ \langle \Psi | T_0 | \Psi \rangle \mid T_0 \in \mathfrak{B}(Q), 0 \leq T_0 \leq I_Q \right. \\ & \quad \left. \sigma = \sum_{j \neq k} p_{jk} |j\rangle\langle j| \otimes |k\rangle\langle k|, 0 \leq \sigma \leq I \right. \\ & \quad \left. T_0 + \sigma \in \text{SEP}, \text{Tr} T_0 + \sigma \leq \alpha d_A d_B \right\} \\ &= \max \left\{ \langle \Psi | T_0 | \Psi \rangle \mid T_0 \in \mathfrak{B}(Q), \right. \\ & \quad \left. \chi(T_0) \leq \alpha d_A d_B, 0 \leq T_0 \leq I_Q \right\} \end{aligned} \quad (70)$$

□

### B. Properties of a function $\chi(\rho)$

In the previous subsection, we saw that  $\bar{S}_{\alpha, \text{Sep}}(|\Psi\rangle)$  gave an upper bound on  $S_{\alpha, \text{Sep}}(|\Psi\rangle)$ , and we can replace the separability condition of POVM in its definition by a condition in terms of a function  $\chi(\rho)$  defined as Eq.(67). For the purpose of further reduction of an upper bound, we give several important properties of  $\chi(\rho)$  which we will use in the next subsection.

First,  $\chi(\rho)$  is closely related to an entanglement measure so called the robustness of entanglement [75], [76]. The robustness of entanglement is defined as

$$R_{s(g)}(\rho) \stackrel{\text{def}}{=} \inf \{ \text{Tr} \sigma : \sigma + \rho \in \text{SEP}, \sigma \in C \}, \quad (71)$$

where  $C$  is  $\text{SEP}$  for  $R_s(\rho)$  (the separable robustness of entanglement), and  $\mathcal{P}_+(\mathcal{H}_{AB})$  for  $R_g(\rho)$  (the global robustness of entanglement), respectively. By the definition, they satisfy  $R_g(\rho) \leq R_s(\rho)$ . It is also known that for a pure state  $|\Psi\rangle = \sum_i \sqrt{\lambda_i} |ii\rangle$ ,

$$R_s(|\Psi\rangle\langle\Psi|) = R_g(|\Psi\rangle\langle\Psi|) = \sum_{j \neq k} \sqrt{\lambda_j \lambda_k}. \quad (72)$$

Generally,  $\chi(\rho)$  gives an upper bound for  $R_s(\rho)$  as follows:

*Lemma 12:* For all  $\rho \in \mathcal{P}_+(Q)$ ,

$$R_s(\rho/\text{Tr}\rho) \leq \chi(\rho)/\text{Tr}\rho - 1. \quad (73)$$

*Proof:* By the definition, we have

$$\chi(\rho/\text{Tr}\rho) = \chi(\rho)/\text{Tr}\rho. \quad (74)$$

Suppose  $\sigma$  attains the minimum of  $\chi(\rho/\text{Tr}\rho)$ . Then, since  $\sigma$  is separable,

$$\begin{aligned} R_s(\rho/\text{Tr}\rho) &\leq \chi(\rho/\text{Tr}\rho) - 1 \\ &= \chi(\rho)/\text{Tr}\rho - 1. \end{aligned}$$

□

Moreover, for a pure state, we can prove the equality of Eq.(73); that is,  $\chi(|\Psi\rangle\langle\Psi|)$  is nothing but equal to the robustness of entanglement  $R_{s(g)}(|\Psi\rangle\langle\Psi|)$  except a constant term:

*Lemma 13:* For a non-normalized state  $|\Psi\rangle = \sum_i a_i |ii\rangle \in Q$ ,

$$\chi(|\Psi\rangle\langle\Psi|) = \sum_{j \neq k} |a_j| |a_k|. \quad (75)$$

Thus, for a normalized pure state  $|\Psi\rangle \in Q$ ,

$$\chi(|\Psi\rangle\langle\Psi|) - 1 = R_s(|\Psi\rangle\langle\Psi|) = R_g(|\Psi\rangle\langle\Psi|) \quad (76)$$

*Proof:* First, we assume  $|\Psi\rangle \in Q$  to be a pure state. Since the Schmidt coefficients of  $|\Psi\rangle$  are  $\{|a_i|\}_{i=1}^{d_A}$ , Lemma 12 and Eq.(72) guarantee

$$R_s(|\Psi\rangle\langle\Psi|) = \sum_{j \neq k} |a_j| |a_k| \leq \chi(|\Psi\rangle\langle\Psi|) - 1. \quad (77)$$

We define a new basis  $\{|\tilde{i}\rangle\}_{i=1}^{d_A}$  of  $\mathcal{H}_A$  so that  $|\Psi\rangle$  can be written down as  $|\Psi\rangle = \sum_i |a_i| |\tilde{i}\tilde{i}\rangle$ . We also define  $T_1$  and  $\sigma$  as

$$T_1 \stackrel{\text{def}}{=} |a\rangle\langle a| \otimes |b\rangle\langle b| \quad (78)$$

$$\sigma \stackrel{\text{def}}{=} \sum_{j \neq k} |a_j| |a_k| |\tilde{j}\tilde{j}\rangle\langle\tilde{j}\tilde{j}| \otimes |k\rangle\langle k|, \quad (79)$$

where  $|a\rangle \stackrel{\text{def}}{=} \sqrt{|a_i|} |\tilde{i}\rangle$  and  $|b\rangle \stackrel{\text{def}}{=} \sqrt{|a_i|} |\tilde{i}\rangle$ . Straightforward calculations yield

$$|\Psi\rangle\langle\Psi| + \sigma = \Gamma(T_1) \in \text{SEP}. \quad (80)$$

Thus, the definition of  $\chi(\rho)$  implies

$$\chi(|\Psi\rangle\langle\Psi|) \leq \sum_{j \neq k} |a_j| |a_k| + 1 = \sum_{jk} |a_j| |a_k|. \quad (81)$$

From the above inequalities and Eq.(77), an arbitrary normalized pure state  $|\Psi\rangle \in Q$  satisfies

$$\chi(|\Psi\rangle\langle\Psi|) = \sum_{jk} |a_j| |a_k|. \quad (82)$$

Finally, by Eq.(74), we can conclude the above equality hold for all non-normalized pure states  $|\Psi\rangle \in Q$ , too. □

By using Lemma 12 and Lemma 13, we can prove that for a general mixed  $\rho \in \mathcal{P}_+(Q)$ ,  $\chi(\rho)$  is derived by just a convex-roof extension from  $\chi(|\Psi\rangle\langle\Psi|)$ , which has analytic formula Eq.(75):

*Lemma 14:* For  $\rho \in \mathcal{P}_+(Q)$ ,

$$\chi(\rho) = \min_{\{p_i, |\Psi_i\rangle\}} \left\{ \sum_i p_i \chi(|\Psi_i\rangle\langle\Psi_i|) \mid \rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i| \right\}. \quad (83)$$

*Proof:* We first prove

$$\chi(\rho) \leq \min_{\{p_i, |\Psi_i\rangle\}} \left\{ \sum_i p_i \chi(|\Psi_i\rangle\langle\Psi_i|) \mid \rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i| \right\}. \quad (84)$$

Suppose  $\rho$  can be decomposed as  $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$ , and  $\sigma_i$  attains the minimum of  $\chi(|\Psi_i\rangle\langle\Psi_i|)$ ; that is,  $|\Psi_i\rangle\langle\Psi_i| + \sigma_i \in SEP$ ,  $\chi(|\Psi_i\rangle\langle\Psi_i|) = \text{Tr}(|\Psi_i\rangle\langle\Psi_i| + \sigma_i)$ , and  $\sigma_i$  also satisfies the remaining conditions. Then, by defining  $\sigma \stackrel{\text{def}}{=} \sum_i p_i \sigma_i$ , we have

$$\rho + \sigma = \sum_i p_i (|\Psi_i\rangle\langle\Psi_i| + \sigma_i) \in SEP. \quad (85)$$

It is also easy to check that  $\sigma$  satisfies the remaining conditions related to  $\chi(\rho)$ . Hence, we have  $\chi(\rho) \leq \sum_i p_i \chi(|\Psi_i\rangle\langle\Psi_i|)$ . Therefore, the inequality (84) holds.

Second, we prove

$$\chi(\rho) \geq \min_{\{p_i, |\Psi_i\rangle\}} \left\{ \sum_i p_i \chi(|\Psi_i\rangle\langle\Psi_i|) \mid \rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i| \right\}. \quad (86)$$

Suppose  $\sigma_{op}$  is optimal for  $\chi(\rho)$ . Then, since  $\rho + \sigma_{op} \in SEP$ , there exists an ensemble of pure states  $\{p_k, |\xi_k\rangle\}_k$  such that

$$\rho + \sigma_{op} = \sum_k p_k |\xi_k\rangle\langle\xi_k| \quad (87)$$

Since  $\rho \in \mathcal{P}_+(Q)$ ,  $Q = \text{span}\{|ii\rangle\}_{i=1}^{d_A}$ , and  $\sigma_{op}$  can be written down as  $\sigma_{op} = \sum_{i \neq j} q_{ij} |i\rangle\langle i| \otimes |j\rangle\langle j|$ , we can see that either  $|\xi_k\rangle \in Q$  or there exist  $i \neq j$  satisfying  $|\xi_k\rangle \propto |ij\rangle$ . Thus, we can write  $\rho + \sigma_{op}$  as

$$\begin{aligned} \rho + \sigma_{op} &= \sum_k p_k \left\{ \left( \sum_i |ii\rangle\langle ii| \right) |\xi_k\rangle\langle\xi_k| \left( \sum_i |ii\rangle\langle ii| \right) \right. \\ &\quad \left. + \sum_{i \neq j} (|ij\rangle\langle ij|) |\xi_k\rangle\langle\xi_k| (|ij\rangle\langle ij|) \right\} \end{aligned} \quad (88)$$

Hence, defining  $|\Psi_k\rangle$  as

$$|\Psi_k\rangle = \left( \sum_i |ii\rangle\langle ii| \right) |\xi_k\rangle, \quad (89)$$

we derive  $\rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$ . Then, we can evaluate  $\text{Tr}|\xi_k\rangle\langle\xi_k|$  as

$$\begin{aligned} &\text{Tr}|\xi_k\rangle\langle\xi_k| \\ &= \text{Tr}\Gamma(|\xi_k\rangle\langle\xi_k|) \\ &= \text{Tr} \left( |\Psi_k\rangle\langle\Psi_k| + \left( \sum_{i \neq j} |ij\rangle\langle ij| |\xi_k\rangle\langle\xi_k| |ij\rangle\langle ij| \right) \right) \\ &\geq \chi(|\Psi_k\rangle\langle\Psi_k|), \end{aligned} \quad (90)$$

where we used the fact  $\Gamma(|\xi_k\rangle\langle\xi_k|) \in SEP$  in the third line. Thus, we can evaluate  $\chi(\rho)$  as follows:

$$\begin{aligned} \chi(\rho) &= \text{Tr}(\rho + \sigma_{op}) \\ &= \sum_k p_k \text{Tr}|\xi_k\rangle\langle\xi_k| \\ &\geq \sum_k p_k \chi(|\Psi_k\rangle\langle\Psi_k|), \end{aligned} \quad (91)$$

where we used Eq.(87) in the second line and the inequality (90) in the third line. The above inequality guarantees that the inequality (86) holds. From the inequalities (86) and (84), Eq.(83) holds.  $\square$

As the next step, for an operator  $\rho \in \mathcal{P}_+(Q)$ , we define a new function  $\chi'(\rho)$  as

$$\chi'(\rho) = \sum_{ij} |\beta_{ij}|, \quad (92)$$

where the coefficients  $\{\beta_{ij}\}_{ij}$  are defined as  $\rho = \sum_{ij} \beta_{ij} |ii\rangle\langle jj|$ . Then, we can show  $\chi'(\rho)$  is a lower bound of  $\chi(\rho)$ :

*Lemma 15:* For  $\rho \in \mathcal{P}_+(Q)$ ,

$$\chi(\rho) \geq \chi'(\rho). \quad (93)$$

Moreover, if  $\text{rank}\rho = 1$ , the equality holds.

*Proof:* Suppose  $\rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$  is a decomposition which attains  $\chi(\rho)$ , and  $\{a_i^{(k)}\}_{ik}$  are coefficients defined as  $a_i^{(k)} \stackrel{\text{def}}{=} \langle ii|\Psi_k\rangle$ . Then, we can evaluate  $\chi(\rho)$  as follows:

$$\begin{aligned} \chi(\rho) &= \sum_k p_k \chi(|\Psi_k\rangle\langle\Psi_k|) \\ &= \sum_k p_k \sum_{ij} |a_i^{(k)} a_j^{(k)}| \\ &= \sum_k p_k \sum_{ij} |\langle ii|\Psi_k\rangle\langle\Psi_k|jj\rangle| \\ &\geq \sum_{ij} \left| \sum_k p_k \langle ii|\Psi_k\rangle\langle\Psi_k|jj\rangle \right| \\ &= \sum_{ij} |\langle ii|\rho|jj\rangle| \\ &= \chi'(\rho), \end{aligned}$$

where we used Eq.( 83) in the first line. Moreover, when  $\text{rank}\rho = 1$ , we can easily see  $\chi(\rho) = \chi'(\rho)$  from Lemma 13.  $\square$

### C. Proof of Theorem

In this subsection, by using lemmas derived in the previous subsections, we complete a proof of Theorem 5.

First, by defining a new function  $\overline{S}'_\alpha(|\Psi\rangle)$  as

$$\begin{aligned} &\overline{S}'_\alpha(|\Psi\rangle) \\ &= \max_T \{ \langle\Psi|T|\Psi\rangle \mid T \in \mathfrak{B}(Q), 0 \leq T \leq I_Q, \chi'(\rho) \leq \alpha d_A d_B \}, \end{aligned} \quad (94)$$

The following inequality follows from Lemma 11 and Lemma 15:

$$\overline{S}'_\alpha(|\Psi\rangle) \geq \overline{S}_{\alpha, SEP}(\Psi). \quad (95)$$

Now, in the definition of  $\overline{S}'_\alpha(\Psi)$ , all related operators are spanned by  $\{|ii\rangle\}_{i=1}^{d_A}$ , and a condition related to separability no more appears. Therefore, we have the following lemma

*Lemma 16:*

$$\overline{S}'_\alpha(|\Psi\rangle) = \max_T \{ \langle \psi | T | \psi \rangle | T \in \mathfrak{B}(\mathcal{H}_A), 0 \leq T \leq I_A, \sum_{ij} |\langle i | T | j \rangle| \leq \alpha d_A d_B \}, \quad (96)$$

where  $|\psi\rangle$  is defined as  $|\psi\rangle \stackrel{\text{def}}{=} \sum_i \sqrt{\lambda_i} |i\rangle$  with the Schmidt coefficients  $\{\lambda_i\}_i$  of  $|\Psi\rangle$ .

Moreover, we can restrict a POVM element  $T$  to a real operator.

*Lemma 17:*

$$\overline{S}'_\alpha(|\Psi\rangle) = \max_T \{ \langle \psi | T | \psi \rangle | T \in \mathfrak{B}(\mathcal{H}_A), 0 \leq T \leq I_A, T = \text{Re}T, \sum_{ij} |\langle i | T | j \rangle| \leq \alpha d_A d_B \}, \quad (97)$$

where  $\text{Re}T$  is defined as  $\text{Re}T \stackrel{\text{def}}{=} \sum_{ij} \text{Re}\langle i | T | j \rangle |i\rangle\langle j|$ .

*Proof:* Suppose  $T$  is an optimal operator attaining  $\chi'(\rho)$  in Eq.(96). Then,  $\overline{T}$  defined as  $\overline{T} \stackrel{\text{def}}{=} \sum_{ij} \langle i | T | j \rangle |i\rangle\langle j|$  is also optimal and attains  $\chi'(\rho)$ . Thus, defining  $T' \stackrel{\text{def}}{=} T + \overline{T}/2$ , we derive  $T' = \text{Re}T'$ ,  $0 \leq T' \leq I_A$ ,  $\langle \psi | T' | \psi \rangle$ , and

$$\sum_{ij} |\langle i | T' | j \rangle| \leq \frac{1}{2} \left\{ \sum_{ij} |\langle i | T | j \rangle| + \sum_{ij} |\langle i | \overline{T} | j \rangle| \right\} \leq \alpha d_A d_B. \quad (98)$$

Thus, we derive Eq.(97).  $\square$

Now, we can show that  $X_{\sqrt{\alpha d_B}}(|\psi\rangle)$  is an upper bound of  $\overline{S}'_\alpha(\Psi)$ :

*Lemma 18:* For a state  $|\Psi\rangle = \sum_{i=1}^{d_A} \sqrt{\lambda_i} |ii\rangle \in \mathcal{H}_{AB}$  and a state  $|\psi\rangle = \sum_{i=1}^{d_A} \sqrt{\lambda_i} |i\rangle \in \mathcal{H}_A$ ,

$$\overline{S}'_\alpha(|\Psi\rangle) \leq X_{\sqrt{\alpha d_B}}(|\psi\rangle). \quad (99)$$

*Proof:* Observing that for  $x_i \in \mathbb{R}$ ,

$$\sum_i |x_i| \leq \epsilon \iff \forall \vec{k} \in \mathbb{Z}_2^d, \sum_i (-1)^{k_i} x_i \leq \epsilon, \quad (100)$$

we can evaluate Eq.(97) as

$$\begin{aligned} \overline{S}'_\alpha(|\Psi\rangle) &= \max_T \{ \langle \psi | T | \psi \rangle | T \in \mathfrak{B}(\mathcal{H}_A), 0 \leq T \leq I_A, T = \text{Re}T, \\ &\quad \forall \vec{k} \in \mathbb{Z}_2^{d_A \times d_A}, \sum_{ij} (-1)^{k_{ij}} \langle i | T | j \rangle \leq \alpha d_A d_B \} \\ &\leq \max_T \{ \langle \psi | T | \psi \rangle | T \in \mathfrak{B}(\mathcal{H}_A), 0 \leq T \leq I_A, T = \text{Re}T, \\ &\quad \forall \vec{k} \in \mathbb{Z}_2^{d_A}, |\langle \phi_{\vec{k}} | T | \phi_{\vec{k}} \rangle| \leq \alpha d_B \}, \end{aligned} \quad (101)$$

where we use the observation  $\langle \phi_{\vec{k}} | T | \phi_{\vec{k}} \rangle = \frac{1}{d_A} \sum_{ij} (-1)^{k_i + k_j} \langle i | T | j \rangle$  in the above inequality. Then, by using the same argument of the proof of Lemma 17, we can remove the restriction of positivity of  $T$  from the maximization in the last line Eq.(101), and derive the inequality (99).  $\square$

Finally, the inequalities (66), (95), (99) yield

$$S_{\alpha, \text{Sep}}(|\Psi\rangle) \leq X_{\sqrt{\alpha d_B}}(|\psi\rangle). \quad (102)$$

Thus, we have succeeded to prove that the optimal success probability of the global hypothesis testing in the last section  $X_{\sqrt{\alpha d_B}}(|\psi\rangle)$  is an upper bound of the optimal success probability of the local hypothesis testing  $S_{\alpha, \text{Sep}}(|\Psi\rangle)$ . Thus, in order to complete the proof of Theorem 5, the remaining task is to show the above inequality is actually an equality. This can be done as follows:

*Proof (Theorem 5):* The inequalities (66), (95) and (99) yield

$$\begin{aligned} S_{\alpha, \text{Sep}}(|\Psi\rangle) &\leq \overline{S}_{\alpha, \text{Sep}}(|\Psi\rangle) \\ &\leq \overline{S}'_\alpha(|\Psi\rangle) \\ &\leq X_{\sqrt{\alpha d_B}}(|\psi\rangle). \end{aligned} \quad (103)$$

Thus, we just need to show that the above three inequalities are actually equalities.

First, we prove the equality

$$\overline{S}'_\alpha(|\Psi\rangle) = X_{\sqrt{\alpha d_B}}(|\psi\rangle). \quad (104)$$

From Theorem 4, an optimal operator  $T \in \mathcal{H}_A$  of  $X_{\sqrt{\alpha d_B}}(|\psi\rangle)$  can satisfy the condition  $\langle i | T | j \rangle \geq 0$  for all  $i$  and  $j$ . Hence, we can calculate as

$$\begin{aligned} \sum_{ij} |\langle i | T | j \rangle| &= \sum_{ij} \langle i | T | j \rangle \\ &= d_A \langle \phi_d | T | \phi_d \rangle \\ &\leq \alpha d_A d_B. \end{aligned} \quad (105)$$

Thus, from Lemma 17, we derive the equality (104).

Second, we prove the equality

$$\overline{S}_{\alpha, \text{Sep}}(|\Psi\rangle) = \overline{S}'_\alpha(|\Psi\rangle). \quad (106)$$

Suppose  $T \in \mathfrak{B}(Q)$  attains the optimum of  $\overline{S}'_\alpha(|\Psi\rangle)$ ; thus,  $T$  satisfies  $0 \leq T \leq I_Q$  and  $\chi'(T) \leq \alpha d_A d_B$ . From Theorem 4, an optimal  $T$  can be written as  $T = |\Phi\rangle\langle\Phi|$ ; that is,  $\text{rank}T = 1$ . Thus, from Lemma 15, we derive  $\chi(T) = \chi'(T)$ . This fact and Lemma 11 guarantee the equality (106).

Finally, we prove the equality

$$S_{\alpha, \text{Sep}}(|\Psi\rangle) = \overline{S}_{\alpha, \text{Sep}}(|\Psi\rangle). \quad (107)$$

First, an optimal operator  $T$  attaining the optimum of Eq.(65) can be written down as

$$T = T_0 + \sigma, \quad (108)$$

where  $T_0$  is an operator attaining the optimum of Eq.(68) and  $\sigma$  is an operator attaining  $\chi(T)$  in terms of Eq.(67).

When  $\eta = 1$  in Theorem 4 for  $d = d_A$  and  $\epsilon = \sqrt{\alpha d_B}$ , we can choose  $T_0$  as  $T_0 = \epsilon_2^2 |11\rangle\langle 11|$ , where  $\epsilon_2 \stackrel{\text{def}}{=} \min\{1, \sqrt{\alpha d_A d_B}/2\}$ . In this case, since  $T_0$  is already separable,  $\chi(T_0) = 1$  and  $\sigma = 0$ . Hence,  $T = T_0 = \epsilon_2^2 |11\rangle\langle 11|$ . Thus,  $I - T$  is a separable operator. Therefore, Eq.(107) holds.

Suppose  $\eta \geq 2$  in Theorem 4 for  $d = d_A$  and  $\epsilon = \sqrt{\alpha d_B}$ . Then,  $T_0$  can be chosen as a normalized pure state. Thus, we have

$$\chi(T_0) = 1 + R_g(T_0). \quad (109)$$

Actually, it is known (in the proof of Lemma 1 of [47]) that we can choose an optimal  $\sigma$  as

$$\sigma = \sum_{j \neq k} \sqrt{\lambda_j \lambda_k} |j\rangle\langle j| \otimes |k\rangle\langle k|, \quad (110)$$

where  $\{\lambda_i\}_{i=1}^{d_A}$  is the Schmidt coefficient of the pure state  $T_0$ . It has already proven that, if  $T$  is defined as  $T = T_0 + \sigma$  by using the above  $\sigma$ ,  $I - T$  is also separable (in the proof of Theorem 2 of [47]). Thus, Eq.(107) holds also in this case. Therefore, Eq.(60) holds.  $\square$

## VII. SUMMARY

In this paper, we have treated a local hypothesis testing whose alternative hypothesis is a bipartite pure state  $|\Psi\rangle$ , and whose null hypothesis is the completely mixed state. As a result, we have analytically derived an optimal type 2 error and an optimal POVM for one-way LOCC POVM (Theorem 1) and Separable POVM (Theorem 3). In particular, in order to derive an analytical solution for Separable POVM, we have proved the equivalence of the local hypothesis testing under Separable POVM and a global hypothesis testing with a composite alternative hypothesis (Section VI), and analytically solved this global hypothesis testing (Section V). Furthermore, for two-way LOCC POVM, we have studied a family of simple three-step LOCC protocols, and have showed that the best protocol in this family has strictly better performance than any one-way LOCC protocol in all the cases where there may exist difference between two-way LOCC POVM and one-way LOCC POVM (Section IV).

Although we restrict ourselves on treating the hypothesis-testing problem in a single-copy scenario in this paper, we are also interested in an extension of our results to problem settings with asymptotically infinite copies of the hypotheses, that is, problem settings like Stein's Lemma [7], and the Chernoff bound [8]. In particular, it is interesting whether the difference of optimal error probabilities under one-way and two-way LOCC survives in the asymptotic extension of the problem. Actually, we have derived new results on this asymptotic extension and are on the way to prepare a manuscript [77].

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## APPENDIX A

### PROOF OF STATEMENTS

#### A. Proof of Corollary 1

The statement about a product state and a maximally entangled state is trivial from Theorem 1, 2 and 3. Thus, we

only give a proof about non-maximally entangled states here. Suppose  $\alpha < 1/d_A d_B$ , that is,  $\epsilon_1 < 1$ . Then, for  $l \geq 2$ ,

$$\begin{aligned} \langle \phi_l | \psi_l \rangle - \epsilon_l &= \left( \frac{\sum_{i=1}^l \sqrt{\lambda_i}}{\sqrt{\sum_{i=1}^l \lambda_i}} - \epsilon_1 \right) / \sqrt{l} \\ &> \left( \frac{\sum_{i=1}^l \sqrt{\lambda_i}}{\sqrt{\sum_{i=1}^l \lambda_i}} - 1 \right) / \sqrt{l} \\ &> 0. \end{aligned} \quad (111)$$

Thus,  $\langle \phi_l | \psi_l \rangle > \epsilon_l$  for  $l \geq 2$ .

In the remaining part, we will prove the statement for separable POVM in the case  $\lambda_1 > \lambda_2$  and in the case  $\lambda_1 = \lambda_2$ , separately.

- 1) In the case  $\lambda_1 > \lambda_2$ , we can prove  $\langle 2 | \phi'_2 \rangle < 0$  as follows:

$$\begin{aligned} &2\sqrt{2}\sqrt{(\lambda_1 + \lambda_2)(1 - c_2^2)}\langle 2 | \phi'_2 \rangle \\ &= (\epsilon_1 - \sqrt{2 - \epsilon_1^2})(\lambda_1 - \lambda_2) \\ &< 0. \end{aligned} \quad (112)$$

Thus,  $\epsilon_2 < \langle \phi_2 | \psi_2 \rangle$ ,  $|\psi_2\rangle \neq |\phi_2\rangle$ , and  $\langle 2 | \phi'_2 \rangle < 0$  guarantee  $\eta = 1$ . Thus, from Theorem 3, we derive  $\beta_{\alpha, \text{sep}} = 1 - \lambda_1 \alpha d_A d_B$  and the optimal POVM is given by  $T(\epsilon_1 | 0) = \alpha d_A d_B |00\rangle\langle 00|$ .

- 2) In the case  $\lambda_1 = \lambda_2$ , there exists a number  $\eta_0$  such that  $\lambda_1 = \dots = \lambda_{\eta_0} > \lambda_{\eta_0+1}$ . In this case, we can easily see  $\eta = \eta_0$ . Then,  $c_{\eta_0} = 1$  guarantees

$$\begin{aligned} \beta_{\alpha, \text{sep}}(|\Psi\rangle) &= 1 - \left( \sum_{i=1}^{\eta_0} \lambda_i \right) \cdot \frac{\alpha d_A d_B}{\eta_0} \\ &= 1 - \lambda_1 \alpha d_A d_B. \end{aligned} \quad (113)$$

We can easily check that a POVM  $T = \alpha d_A d_B |11\rangle\langle 11|$  attains this optimum.

Finally, since the above POVM can be implemented by one-way LOCC, we derive the statement of the corollary.  $\square$

## REFERENCES

- [1] C.W. Helstrom, *Quantum detection and estimation theory* Academic Press, New York, 1976.
- [2] A.S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, North Holland, Amsterdam, 1982.
- [3] H.P. Yuen, R.S. Kennedy, and M. Lax, "Optimum testing of multiple hypotheses in quantum detection theory," *IEEE Trans. Inf. Theory*, IT-21, 125, 1975.
- [4] W.K. Wootters and B.C. Fields, "Optimal state-determination by mutually unbiased measurements," *Ann. Phys.* vol.191, 363, 1989.
- [5] A. Peres and W.K. Wootters, "Optimal detection of quantum information," *Phys. Rev. Lett.* vol.66, 1119, 1991.
- [6] A.S. Holevo, "An analog of the theory of statistical decisions in non-commutative theory of probability," *Trudy Moskov. Mat. Obšč.*, vol. 26, 133, 1972. (English translation is *Trans. Moscow Math. Soc.*, vol.26, 133, 1972.
- [7] F. Hiai and D. Petz, "The proper formula for relative entropy and its asymptotics in quantum probability," *Commun. Math. Phys.*, vol.143, 99, 1991.
- [8] K.M.R. Audenaert, J. Calsamiglia, R. Muñoz-Tapia, E. Bagan, L.I. Masanes, A. Acín, F. Verstraete, "Discriminating States: The Quantum Chernoff Bound," *Phys. Rev. Lett.* vol.98, 160501 2007.
- [9] C.W. Helstrom, "Minimum Mean-Square Error Estimation in Quantum Statistics," *Phys. Lett. A*, vol.25, 101 1967.



- [10] H.P. Yuen, and M. Lax, "Multiple-Parameter Quantum Estimation and Measurement of Nonselfadjoint Observables," *IEEE Trans. Inf. Theory*, vol.19, 740, 1973.
- [11] A.S. Holevo, "Covariant measurements and uncertainty relations," *Reports in Mathematical Physics* vol.16, 385, 1979.
- [12] A.S. Holevo, "On the capacity of quantum communication channel," *Problemy Peredachi Informatsii*, vol.15, 4, 3, 1979. English translation: *Probl. Inf. Transm.* vol.15, 247, 1979.
- [13] A.S. Holevo, "The capacity of the quantum channel with general signal states," *IEEE Trans. Inf. Theory*, vol.44, 269, 1998.
- [14] B. Schumacher, M.D. Westmoreland, "Sending classical information via noisy quantum channels," *Phys. Rev. A*, vol.54, 2614, 1996.
- [15] M. Hayashi, *Asymptotic Theory Of Quantum Statistical Inference: Selected Papers*, World Scientific Pub Co Inc, 2005.
- [16] M. Hayashi, *Quantum Information: An Introduction*, Springer-Verlag, 2006
- [17] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, W. K. Wootters "Purification of Noisy Entanglement and Faithful Teleportation via Noisy Channels" *Phys. Rev. Lett.* vol.76, 722, 1996.
- [18] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, W. K. Wootters, "Mixed-state entanglement and quantum error correction" *Phys. Rev. A* vol.54, 3824, 1996.
- [19] S. Virmani and M.B. Plenio, "An introduction to entanglement measures," *Quant. Inf. Comp.* vol.7, 1, 2007.
- [20] R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, "Quantum entanglement," *Rev. Mod. Phys.* vol.81, 865, 2009.
- [21] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, "Quantum nonlocality without entanglement," *Phys. Rev. A*, vol.59, no. 2, pp. 1070-1091, 1999.
- [22] J. Walgate, A. J. Short, L. Hardy, and V. Vedral, "Local Distinguishability of multipartite orthogonal quantum states," *Phys. Rev. Lett.*, vol. 85, 4972, 2000.
- [23] S. Virmani, M. F. Sacchi, M. B. Plenio, and D. Markham, "Optimal local discrimination of two multipartite pure states," *Phys. Lett. A*, vol. 288, 62, 2001.
- [24] S. Ghosh, G. Kar, A. Roy, A. Sen(De), and U. Sen, "gDistinguishability of Bell states," *Phys. Rev. Lett.*, vol.87, 277902, 2001.
- [25] B. M. Terhal, D. P. DiVincenzo, and D. W. Leung, "Hiding bits in Bell states," *Phys. Rev. Lett.*, vol.86, 5807, 2001.
- [26] D. P. DiVincenzo, D. W. Leung, and B. M. Terhal, "gQuantum data hiding," *IEEE Trans. Inf. Theory*, vol.48, 580, 2002.
- [27] T. Eggeling and R. F. Werner, "Hiding Classical Data in Multipartite Quantum States," *Phys. Rev. Lett.*, vol. 89, 097905, 2002.
- [28] J. Walgate and L. Hardy, "Nonlocality, asymmetry, and distinguishing bipartite states," *Phys. Rev. Lett.*, vol.89, 147901, 2002.
- [29] Y.-X. Chen and D. Yang, "Optimally conclusive discrimination of nonorthogonal entangled states by local operations and classical communications," *Phys. Rev. A*, vol.65, 022320, 2002.
- [30] P. Badziag, M. Horodecki, A. Sen De, U. Sen, "Locally accessible information: How much can the parties gain by cooperating?" *Phys. Rev. Lett.* vol.91, 117901, 2003.
- [31] M. Hillery and J. Mimihi, "Distinguishing two-qubit states using local measurements and restricted classical communication," *Phys. Rev. A*, vol. 67, 042304, 2003.
- [32] S. Virmani, M.B. Plenio, "Construction of extremal local positive-operator-valued measures under symmetry," *Phys. Rev. A* vol.67, 062308 (2003)
- [33] A. Chefles, "Condition for unambiguous state discrimination using local operations and classical communication," *Phys. Rev. A*, vol. 69, 050307(R), 2004.
- [34] H. Fan, "Distinguishability and indistinguishability by local operations and classical communication," *Phys. Rev. Lett.*, vol. 92, 177905, 2004.
- [35] Z. Ji, H. Cao, and M. Ying, "Optimal conclusive discrimination of two states can be achieved locally," *Phys. Rev. A*, vol. 71, 032323, 2005.
- [36] P. Hayden and C. King, "Correcting quantum channels by measuring the environment," *Quant. Inf. Comp.*, vol. 5, 156, 2005
- [37] M. Nathanson, "Distinguishing bipartite orthogonal states using LOCC: best and worst cases," *J. Math. Phys.* vol. 46, 062103, 2005.
- [38] J. Watrous, "Bipartite subspaces having no bases distinguishable by local operations and classical communication," *Phys. Rev. Lett.*, vol. 95, 080505, 2005.
- [39] M. Hayashi, D. Markham, M. Murao, M. Owari, and S. Virmani, "Bounds on multipartite entangled orthogonal state discrimination using local operations and classical communication," *Phys. Rev. Lett.*, vol. 96, 040501, 2006.
- [40] M. Hayashi, K. Matsumoto, Y. Tsuda, "A study of LOCC-detection of a maximally entangled state using hypothesis testing," *J. Phys. A: Math. Gen.* vol.39, 14427, 2006.
- [41] M. Owari and M. Hayashi, "Local copying and local discrimination as a study for nonlocality of a set of states," *Phys. Rev. A* vol.74, 032108, 2006.
- [42] Y. Ogata, "Local distinguishability of quantum states in infinite-dimensional systems," *J. Phys. A: Math. Gen.*, vol. 39, 3059, 2006.
- [43] R. Duan, Y. Feng, Z. Ji, M. Ying, "Distinguishing Arbitrary Multipartite Basis Unambiguously Using Local Operations and Classical Communication," *Phys. Rev. Lett.* vol.98, 230502, 2007.
- [44] M.-Y. Ye, W. Jiang, P.-X. Chen, Y.-S. Zhang, Z.-W. Zhou, and G.-C. Guo, "Local distinguishability of orthogonal quantum states and generators of SU(N)" *Phys. Rev. A* vol.76, 032329, 2007.
- [45] H. Fan, "Distinguishing bipartite states by local operations and classical communication," *Phys. Rev. A* vol.75, 014305, 2007.
- [46] W. Matthews and A. Winter, "On the Chernoff Distance for Asymptotic LOCC Discrimination of Bipartite Quantum States," *Commun. Math. Phys.* vol. 285, 161, 2008.
- [47] M. Owari, M. Hayashi, "Two-way classical communication remarkably improves local distinguishability," *New J. of Phys.*, Vol. 10, 013006, 2008.
- [48] Y. Xin, R. Duan, "Local distinguishability of orthogonal  $2 \otimes 3$  pure states," *Phys. Rev. A* vol.77, 012315, 2008.
- [49] A. Hayashi, T. Hashimoto, M. Horibe, "State discrimination with error margin and its locality," *Phys. Rev. A* vol.78, 012333, 2008.
- [50] Y. Ishida, T. Hashimoto, M. Horibe, A. Hayashi, "Locality and nonlocality in quantum pure-state identification problems," *Phys. Rev. A* vol.78, 012309, 2008.
- [51] S.M. Cohen, "Almost every set of  $N \geq d + 1$  orthogonal states on  $d^{\otimes n}$ ," *Phys. Rev. A* vol.77, 060309(R), 2008.
- [52] H. Sugimoto, T. Hashimoto, M. Horibe, A. Hayashi, "Discrimination with error margin between two states - Case of general occurrence probabilities," *Phys. Rev. A* vol.80, 052322, 2009.
- [53] R. Duan, Y. Feng, Y. Xin, M. Ying, "Distinguishability of Quantum States by Separable Operations," *IEEE Trans. Inf. Theory*, vol.55, 1320, 2009.
- [54] S. Bandyopadhyay and J. Walgate, "Local distinguishability of any three quantum states," *J. Phys. A: Math. Gen.* vol.42, 072002, 2009.
- [55] M. Hayashi, "Group theoretical study of LOCC-detection of maximally entangled state using hypothesis testing," *New J. of Phys.* vol.11, 043028, 2009.
- [56] W. Jiang, X.-J. Ren, X. Zhou, Z.-W. Zhou, and G.-C. Guo "Subspaces without locally distinguishable orthonormal bases," *Phys. Rev. A* vol.79, 032330, 2009.
- [57] M. Nathanson, "Testing for a pure state with local operations and classical communication," *J. Math. Phys.* vol.51, 042102, 2010.
- [58] T. Ogawa, and H. Nagaoka, "Strong Converse and Stein's Lemma in Quantum Hypothesis Testing," *IEEE Trans. Inf. Theory*, vol. 46, 2428, 2000.
- [59] M. Hayashi, "Optimal sequence of quantum measurements in the sense of Stein's lemma in quantum hypothesis testing," *J. Phys. A: Math. and Gen.*, vol.35, 10759, 2002.
- [60] T. Ogawa, and M. Hayashi, "On Error Exponents in Quantum Hypothesis Testing," *IEEE Trans. Inf. Theory*, vol.50, 1368, 2004.
- [61] M. Hayashi, B.-S. Shi, A. Tomita, K. Matsumoto, Y. Tsuda, Y.-K. Jiang, "Hypothesis testing for an entangled state produced by spontaneous parametric down conversion," *Phys. Rev. A*, vol.74, 062321, 2006.
- [62] F. Hiai, M. Mosonyi, and T. Ogawa, "Large Deviations and Chernoff Bound for Certain Correlated States on the Spin Chain," *J. Math. Phys.*, vol.48, 123301, 2007.
- [63] H. Nagaoka, M. Hayashi, "An Information-Spectrum Approach to Classical and Quantum Hypothesis Testing for Simple Hypotheses," *IEEE Trans. Inf. Theory*, vol.53, 534, 2007.
- [64] F. Hiai, M. Mosonyi, and T. Ogawa, "Error Exponents in Hypothesis Testing for Correlated States on a Spin Chain," *J. Math. Phys.*, vol. 49, 032112, 2008.
- [65] F. Hiai, M. Mosonyi, M. Hayashi, "Quantum hypothesis testing with group symmetry," *J. Math. Phys.* vol.50, 103304, 2009.
- [66] M. Mosonyi, "Hypothesis testing for Gaussian states on bosonic lattices," *J. Math. Phys.* vol.50, 032105, 2009.
- [67] F.G.S.L. Brandao, and M.B. Plenio, "A Generalization of Quantum Stein's Lemma," *Commun. Math. Phys.* vol.295, 791, 2010.
- [68] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, 2000.
- [69] M. J. Donald, M. Horodecki, O. Rudolph, "The uniqueness theorem for entanglement measures," *J. Math. Phys.* vol.43, 4252, 2002.

- [70] R. F. Werner, "Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model," *Phys. Rev. A* vol.40, 4277, 1989.
- [71] E. M. Rains, "A semidefinite program for distillable entanglement," *IEEE Trans. Inf. Theory*, vol.47, 2921, 2001.
- [72] K. G. H. Vollbrecht and R. F. Werner, "Entanglement measures under symmetry," *Phys. Rev. A* vol.64, 062307, 2001.
- [73] M. Hayashi, D. Markham, M. Murao, M. Owari, and S. Virmani, "Entanglement and group symmetries: stabilizer states, symmetric and antisymmetric States," *Phys. Rev. A* vol.77, 012104, 2008.
- [74] T. Hiroshima and M. Hayashi, "Finding a maximally correlated state: Simultaneous Schmidt decomposition of bipartite pure states," *Phys. Rev. A* vol.70, 030302(R), 2004.
- [75] G. Vidal and R. Tarrach, "Robustness of entanglement," *Phys.Rev. A* vol.59, 141, 1999.
- [76] A.W. Harrow and M.A. Nielsen, "How robust is a quantum gate in the presence of noise?" *Phys. Rev. A* vol.68, 012308, 2003.
- [77] M. Owari and M. Hayashi, in preparation.